# The power-free subset problem

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## 1. The sum-free subset problem

Let $A = \{a_1, a_2, \cdots, a_n\} \subset \mathbb{R}$, where $a_i \neq 0$ for every $i \in \{1, \ldots, n\}$. Let $f(A)$ be the largest integer, such that $\exists B \subseteq A$, $|B| = f(A)$ and $x + y \neq z$, for any $x, y, z \in B$. $B$ will be called *sum-free* subset. A triple of elements $x, y, z$, such that $x + y = z$, will be called a *sum-triple*. The following result was obtained by Erdős using an ingenious probabilistic argument. In particular, he constructs a sum-free subset probabilistically and shows that its expected size is $|A|/3$.

**Theorem 1** (Erdős, [4]). $\forall A \subset \mathbb{R} : f(A) \geq \frac{1}{3}|A|$.

In other words, for every set of $n$ nonzero real numbers, one is always able to select a subset with at least $\frac{n}{3}$ numbers that is sum-free. Theorem 1 has raised the question whether $\frac{1}{3}$ is the largest possible constant having this property. More formally, what is the true value of $\alpha_+ := \inf\limits_{A \subset \mathbb{R}} \frac{f(A)}{|A|}$? This question had been an object of research for nearly 50 years. Theorem 1 gives that $\alpha_+ \geq \frac{1}{3}$. Several upper bounds were obtained after that. [6] is a good survey on the previous work related to the problem. In 2013, Eberhard, Green and Manners [3] proved that $\frac{1}{3}$ is also an upper bound for $\alpha_+$. In particular, they showed that for every $\epsilon > 0$, there exists a set $A$ of $n$ integers with the following property: for every $A' \subseteq A$ with at least $(\frac{1}{3} + \epsilon)n$ elements, $\exists x, y, z \in A'$, such that $x + y = z$. Therefore, they proved the following theorem.

**Theorem 2** (Eberhard et al., [3]).

$$\alpha_+ = \frac{1}{3}.$$

The same question was considered for arbitrary abelian groups instead of $(\mathbb{R}, +)$ and the corresponding optimal constant was found to be $\frac{2}{7}$ (see [1]). Note also that the product-free problem over $\mathbb{R}$ is equivalent to the sum-free problem since $x * y = z$ iff $\log x + \log y = \log z$.

## 2. The power-free subset problem

Instead of sum-free sets, let us consider *power-free* sets. A triple of elements $x, y, z$, such that $x^y = z$, will be called a *power-triple*. Denote the set of real numbers greater than 1 by $\mathbb{R}_{>1}$. We will look only at sets of numbers in $\mathbb{R}_{>1}$ since this is needed to prove Theorem 3. If $X \subset \mathbb{R}_{>1}$ is a set of $n$ numbers, then let $g(X)$ be the largest integer, such that $\exists Y \subseteq X$, $|Y| = g(X)$ and $Y$ is power-free, i.e., it does not contain a power-triple. We want to obtain $\alpha_{\text{pow}} := \inf\limits_{X \subset \mathbb{R}_{>1}} \frac{g(X)}{|X|}$. Note that the set $\mathbb{R}_{>1}$ endowed with the power operation is not a group. The following lower bound for $\alpha_{\text{pow}}$ was obtained by using the probabilistic argument of Erdős for the set $\{\log x \mid x \in A\}$ with an additional observation utilizing the pigeonhole principle.

**Theorem 3** (Noga Alon, private communication).

$$\alpha_{\text{pow}} \geq \frac{1}{8}.$$

An upper bound for $\alpha_{\text{pow}}$ can be obtained by the following explicit construction. Consider $A = A_1 \cup A_2$, where

$$A_1 = \{d^k, d^{2k}, \ldots, d^{nk}\}$$

and

$$A_2 = \{a^{d^k}, a^{d^{2k}}, \ldots, a^{d^{nk}}\},$$

for some integers $a, d$ and $k$ for which all of the $2n$ listed numbers in $A$ are different and greater than 1. Then, it is not difficult to show that every $A' \subset A$ with $n+2$ numbers is not power-free, i.e., it contains a power-triple. The latter implies that $\alpha_{\text{pow}} \leq \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$.

We want to improve the given lower and upper bounds for $\alpha_{pow}$. Finding the exact value of $\alpha_{pow}$ is equivalent to finding lower and upper bounds that are equal.

**Question 1.** What is the value of $\alpha_{\text{pow}}$?

It would be interesting even to know how $\alpha_{\text{pow}}$ and $\alpha_+$ compare.

2.1. **Possible approaches.** Improving the lower and the upper bounds would require different approaches. To improve the lower bound, one must show that for all $A$, we are always able to find a subset of $A$, which is power-free and has at least $c|A|$ elements, for some $c > \frac{1}{8}$. One may try to use Bourgain's idea [2] to reinterpret Erdős' proof in the language of Fourier analysis. Previous results in additive combinatorics concerning free-sum subsets could be eventually used, as well (see the notes of Granville [5]). To improve the upper bound, we need to construct an example of a particular set $A$, such that each $A' \subset A$, with $|A'| \geq d|A|$, for some $d \leq \frac{1}{2}$, is not power-free. One idea is to try establishing the conjecture below.

**Conjecture 1.** Let $A = \{a_1, \ldots, a_n\} \in 2^{\mathbb{R}}$ be a finite set of $n$ real numbers. Then, there exists a set $B = \{b_1, \ldots, b_n\} \in 2^{\mathbb{R}_{>1}}$, such that if $a_{i_1} + a_{i_2} = a_{i_3}$, for some $1 \leq i_1, i_2, i_3 \leq n$, then $b_{j_1}^{b_{j_2}} = b_{j_3}$, where $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\}$.

If this conjecture holds, then one may take $A$ to be the set from the proof of Theorem 2. Recall that each of its subsets of size $(\frac{1}{3} + \epsilon)|A|$ contains a sum-triple. Then, the conjecture would give us that there exists a set $B$ with elements in $\mathbb{R}_{>1}$, such that each sum-triple in $A$ has a corresponding power-triple in $B$. This would imply that $\alpha_{\text{pow}} \leq \frac{1}{3}$.

In addition, one may consider the set $S = \{2, 3, 4, 5, 6, 8, 10\}$ attributed to Klarner [4], which gives $\alpha_+ \leq \frac{3}{7}$, and try to come up with a corresponding set of 7 numbers in $\mathbb{R}_{>1}$, having a power-triple at the place of each sum-triple in $S$. Such a set would give a new upper bound of $\frac{3}{7}$ for the power-free problem. Showing that such a set does not exist will refute Conjecture 1.

According to Mark Lewko (private communication), a modification of the Bourgain's approach [2] can give that for any set $S$ of $n$ real numbers, there always exists a power-free subset of $S$ with $n/8 + cn^{1/2}/\log n$ elements, for some small fixed c.

## References

[1] Alon, N. and Kleitman, D.J., 1990. Sum-free subsets. A tribute to Paul Erdos, pp.13-26.
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[4] Erdős, P., 1965. Extremal problems in number theory. In Proc. Sympos. Pure Math (pp. 181-189).
[5] https://dms.umontreal.ca/ andrew/PDF/AddComb.pdf
[6] Lewko, M., 2013. The Eberhard-Green-Manners Sum-free Subset Theorem and what is Sum-free(13)?, available at https://lewko.wordpress.com/2013/02/19/the-eberhard-green-manners-sum-free-subset-theorem-and-what-is-sum-free13.