Digraphs with exactly one Eulerian tour

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Abstract

We give two combinatorial proofs of the fact that the number of loopless directed graphs (digraphs) with n non-isolated vertices and with exactly one Eulerian tour up to a cyclic shift is $\frac{1}{2}(n-1)!C_n$, where C_n denotes the n-th Catalan number. We construct a bijection with a set of labeled rooted plane trees and with a set of valid parenthesis arrangements.

1 Introduction and main facts

Richard Stanley has a list containing nearly 250 problems and facts, for which he asks of combinatorial proofs [1]. This work describes two such proofs for one of the problems in

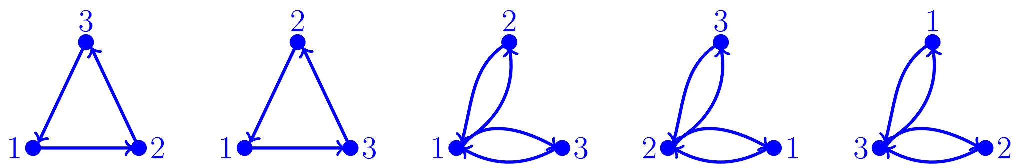

Figure 1: The five digraphs with three vertices and a unique Eulerian tour.

the list, namely Problem 199, without a known combinatorial proof. First, we recall some definitions following [2].

A (finite) directed graph (or digraph) D consists of a vertex set $V = \{v_1, \ldots, v_n\}$ and an edge set $E = \{e_1, \ldots, e_q\}$, together with a function $\phi : V \to V$ determining the direction of each edge. If $\phi(e) = (u, v)$, then we think of e as an arrow from u to v. We will call u initial vertex and v final vertex. The outdegree of a vertex v, denoted $\operatorname{outdeg}(v)$, is the number of edges of D with initial vertex v. Similarly, the indegree of v, denoted $\operatorname{indeg}(v)$, is the number of edges of D with final vertex v. A loop is an edge e for which $\phi(e) = (v, v)$ for some vertex v. A digraph is balanced if $\operatorname{indeg}(v) = \operatorname{outdeg}(v)$ for each of its vertices v. Let $[n] \coloneqq \{1, 2, \ldots, n\}$. An oriented path in a digraph D is a sequence of vertices $v_1, \ldots, v_m$, where (v_i, v_{i+1}) is an edge of D for each $i \in [m-1]$. If the vertices $v_1, \ldots, v_m$ are all different, then we call the path *simple*. If we have a simple path $v_1, \ldots, v_{m-1}$, then we call the path $v_1, \ldots, v_{m-1}, v_1$ an *oriented simple cycle*.

Definition 1. An *Eulerian tour* in a directed graph D is a sequence of vertices $a_1 a_2 \cdots a_k$ such that (a_1, a_2), $(a_2, a_3), \cdots, (a_{k-1}, a_k)$, (a_k, a_1) are all the distinct directed edges of D.

Any cyclic shift $a_i a_{i+1} \cdots a_k a_1 \cdots a_{i-1}$ of an Eulerian tour is also an Eulerian tour and we will say that these tours are *equivalent up to a cyclic shift*.

Definition 2. A *uni-Eulerian digraph* is a digraph which has no isolated vertices and contains exactly one Eulerian tour (and its equivalents under cyclic shifts).

Our goal is to prove the following claim.

Theorem 3. *If A_n is the set of loopless uni-Eulerian digraphs on the vertex set $[n]$, then $|A_n| = \frac{1}{2}(n-1)!C_n$ (sequence A102693 in the OEIS [5]), where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the n-th Catalan number.*

For example, $|A_3| = 5$. Indeed, there are two such digraphs that look like triangles and three that consist of two 2-cycles with a common vertex (see Figure 1). The related OEIS sequence, A102693, was created by Richard Stanley. As a reference, he points out to an unpublished work of him. Thus, we can assume that Theorem 3 was first proved there.

It is not difficult to show that every uni-Eulerian graph must be connected and balanced [2, Theorem 10.1]. The BEST theorem that we recall below gives us a formula for the total number of Eulerian tours in a digraph. In order to understand this result, one should be familiar with the term *oriented tree* (see Figure 2). An oriented tree with root v is a finite digraph T with v as one of its vertices, such that there is a unique directed path from any other vertex of T to v. This means that the underlying undirected graph (after we erase all the arrows of the edges of T) is a tree.

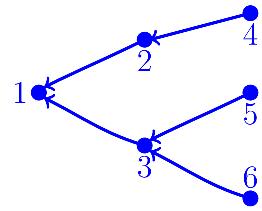

Figure 2: Example of an oriented tree.

Theorem 4 (BEST theorem, [2]). *Let D be a connected balanced digraph with vertex set V. Fix an edge e in D and let v be the initial vertex of that edge. Let $\tau(D, v)$ denote the number of oriented (spanning) subtrees of D with root v, and let $\epsilon(D, e)$ denote the number of Eulerian tours of D starting with the edge e. Then*

$$\epsilon(D, e) = \tau(D, v) \prod_{u \in V} (\text{outdeg}(u) - 1)!$$

Corollary 5 (from Theorem 4). *A digraph $D \in A_n$ if and only if*

1. *For every vertex v, D has exactly one oriented (spanning) subtree with root v.*
2. *The outdegree of an arbitrary vertex of D is 1 or 2.*

Note that if the two conditions of Corollary 5 hold, then $\epsilon(D, e) = 1$ for every edge e of the digraph D and thus D has a unique Eulerian tour. Using Corollary 5, we will characterize the digraphs in A_n by two other conditions that will be used later.

Lemma 6. *A digraph $D \in A_n$ if and only if*

i) There exists a unique oriented simple path between any two vertices of D.

ii) Every vertex of D is part of exactly one or two simple oriented cycles.

Corollary 7. *If $D \in A_n$, then no pair of cycles in D have an edge in common.*

Proof. Assume C_1 and C_2 are two different cycles in D, which have the edge (u, v) in common. Then, we will have at least two oriented paths from v to u in D — one following C_1 and another one following C_2. This is a contradiction with Corollary 5. □

Proof of Lemma 6. [First part: $D \in A_n \implies$ conditions $i)$ and $ii)$] Let $D \in A_n$. Let u and v be two arbitrary vertices in D. By condition (1) of Corollary 5, there exists exactly one oriented spanning subtree T_u of D with root u. We know that v has to be a vertex of T_u and that there exists a unique simple path $\mathcal{P}$ from v to u in T_u, which is a simple oriented path in D. Assume that there exists another path $\mathcal{P}' \neq \mathcal{P}$ between v and u in D. Begin from v and follow $\mathcal{P}'$. Let f' be the first edge in $\mathcal{P}'$ which is not part of $\mathcal{P}$ and let f be the edge in $\mathcal{P}$ with the same initial vertex as f'. If you delete f from T_u and add f' to it, you will obtain a graph T_u'. One can easily see that T_u' is an oriented spanning subtree of D, different from T_u (see Figure 3). This is a contradiction. Thus, we showed that condition $i)$ holds.

By Corollary 7, we know that no pair of cycles in D have an edge in common. Therefore, Corollary 5 implies that each vertex of D can be part of at most two simple oriented cycles since its outdegree is 1 or 2. It remains to show that each vertex of D is part of at least one such cycle. Let v be an arbitrary vertex of D and let (u, v) be an edge of D (such an edge exists since the indegree of v is 1 or 2). We showed that there is a unique oriented simple path between v and u. This path together with the edge (u, v) forms a simple oriented cycle. Thus condition ii) holds.

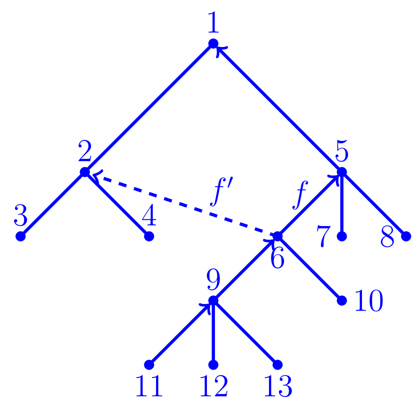

Figure 3: The tree T_u in the first part of the proof of Lemma 6; $u = 1$, $v = 11$, $\mathcal{P} = 11, 9, 6, 5, 1$, $\mathcal{P}' = 11, 9, 6, 2, 1$, $f' = (6, 2)$ and $f = (6, 5)$. Delete f and add f' to obtain another tree T'_u.

[second part: conditions i) and ii) $\Longrightarrow D \in A_n$] Take a digraph D for which conditions i) and ii) hold. We have to show that conditions (1) and (2) from Corollary 5 also hold. Let v be an arbitrary vertex of D. Condition ii) implies that $\text{outdeg}(v) \geq 1$. We will show that $\text{outdeg}(v) < 3$. By Corollary 7 no edge of D can be part of two different simple cycles. Assume that $\text{outdeg}(v) \geq 3$ and let (v, u_1), (v, u_2) and (v, u_3) are three different edges of D. We know that there exist simple paths between u_i and v, for $i = 1, 2, 3$. Thus, v participates in simple cycles through u_i, for $i = 1, 2, 3$ and no two of these simple cycles share and edge. Therefore these three cycles are different. This is a contradiction with condition ii).

It remains to show that condition (1) from Corollary 5 holds. Take an arbitrary vertex v of D. We have a unique oriented simple path from each of the other vertices of D to v. Take the union of these paths. The graph that you will obtain is an oriented spanning subtree T_v of D with root v. Assume that there exists another such subtree T'_v. Then, since $T_v \neq T'_v$, we must have a vertex w in D, for which the unique oriented path from w to v in T'_v is different from the unique oriented path from w to v in T_v. These are two different oriented paths from w to v in D, which is a contradiction. □

2 Bijection with a set of labeled rooted plane trees

We will construct a bijection between the digraphs in A_n and a set of labeled rooted plane trees on $n + 1$ vertices. Let U_n be the set of the unlabeled rooted plane trees with $n + 1$

vertices. It is well-known that $|U_n| = C_n$ (see [3, Theorem 1.5.1]). Let L_n be the set of the labeled rooted plane trees with $n+1$ vertices such that the root is always labeled as 0 and the left-most child of the root is always labeled as 1. We have $|L_n| = (n-1)!|U_n| = (n-1)!C_n$. Finally, let $L'_n \subset L_n$ be the set of the labeled trees in L_n, such that the vertex 2 is in the subtree with root 1.

First, we define a map f over L_n, such that $f(L'_n) = L_n \setminus L'_n$ and $f(f(T)) = T$, i.e., an involution. This shows that $|L'_n| = \frac{|L_n|}{2} = |A_n|$. Then, we define a map $g : L'_n \to A_n$, which is shown to be a bijection. Below, we will denote by $T_{(x,j)}$ the j-th child (from left to right) of the vertex x of a tree $T \in L_n$.

Definition 8. Let $f : L_n \to L_n$ be a map, which switches the places of the subtree with root 1 (excluding 1) and the subtree with root 0 (excluding 0 and the subtree with root 1), for every tree in L_n (see Figure 4). Formally, if $T \in L_n$, then $f(T)$ has the following properties. For every $j \geq 1$:

- If the vertex v is the (j+1)-th child of the vertex 0 in T, then v is the j-th child of vertex 1 in $f(T)$, i.e., $T_{(0,j+1)} = f(T)_{(1,j)}$.
- If the vertex v is the j-th child of the vertex 1 in T, then v is the (j+1)-th child of vertex 0 in $f(T)$, i.e., $T_{(1,j)} = f(T)_{(0,j+1)}$.
- All the other directed edges are left the same for both trees, i.e., $T_{(u,j)} = f(T)_{(u,j)}$ for $u \notin \{0, 1\}$.

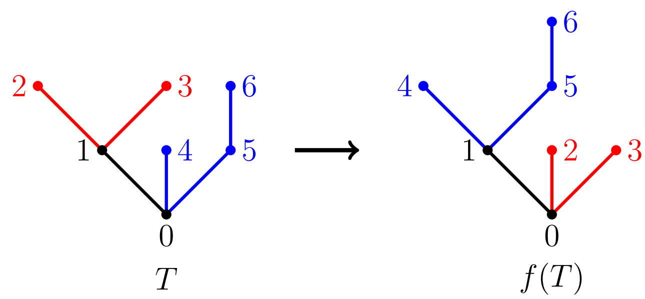

Figure 4: Example of the action of the map f.

It is easy to check that $f(L'_n) = L_n \setminus L'_n$ and $f(f(T)) = T$, i.e., $f^{-1} = f$.

Definition 9. Let $g : L'_n \to A_n$ be a map, such that if $T \in L'_n$, $g(T) = D'$ is a digraph with $V(D') = [n]$ and $E(D')$, such that if x is a vertex of T with r children, then:

1. For every $i \in [1, r)$, $(T_{(x,i)}, T_{(x,i+1)}) \in E(D')$.
2. If $x = 0$, then $(T_{(x,r)}, T_{(x,1)}) \in E(D')$.
3. If $x \neq 0$, then $(T_{(x,r)}, x) \in E(D')$ and $(x, T_{(x,1)}) \in E(D')$.

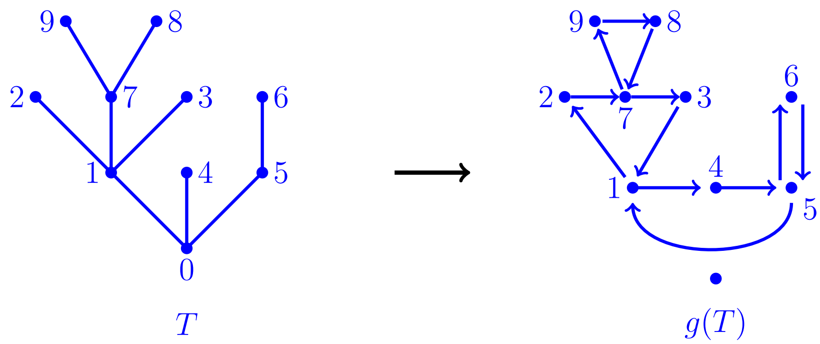

Figure 5: Example of the action of the map g.

Lemma 10. *For every $T \in L'_n$, $g(T) \in A_n$, i.e., $g(T)$ has exactly one Eulerian tour.*

Proof. First, note that every vertex x of $g(T)$ belongs to one or two cycles:

- The cycle where x and its parent from T both belong.
- In case x has children in T, the cycle formed by x and its children in $g(T)$.

By Lemma 6, it remains to show that there exists a unique oriented simple path between any two vertices of $g(T)$, for arbitrary $T \in L'_n$. To see this, observe that if $(v, w) \in E(T)$ and $v, w \neq 0$ or if both v and w are children of the root 0 in T, then we have a unique oriented simple path between v and w in $g(T)$, which is part of a single cycle. For instance, the edge $(1, 3)$ in the graph T shown at Figure 5 corresponds to the oriented simple path 1273 in $g(T)$, whereas the path 514 in $g(T)$ corresponds to the pair of children 4 and 5 of the root of T.

We will show that since we have a unique non-oriented path $\mathcal{P}$ between any two vertices u and v in T, where $u, v \neq 0$, we will also have a unique oriented simple path $\mathcal{P}^{\text{or}}$ between u and v in $g(T)$. If the vertex 0 is part of $\mathcal{P}$, then we must have vertices h_1 and h_2 in T, such that the edges $(h_1, 0)$ and $(0, h_2)$ are part of $\mathcal{P}$ (since $v, w \neq 0$). Hence, h_1 and h_2 are two children of the root 0. Replace the edges $(h_1, 0)$ and $(0, h_2)$ of $\mathcal{P}$ with the unique oriented simple path between h_1 and h_2 in $g(T)$. Replace all the other edges of $\mathcal{P}$ with the corresponding oriented simple paths to obtain $\mathcal{P}^{\text{or}}$. For example, the unique path $\mathcal{P}$ between 3 and 8 in the graph T on Figure 5 is comprised of the edges $(3, 1)$, $(1, 7)$, $(7, 8)$. The oriented paths corresponding to these edges are 31, 127 and 798, respectively. The union of these paths, namely 312798, gives the unique path $\mathcal{P}^{\text{or}}$ between 3 and 8 in $g(T)$. Another example is the path 71056 in T, which transforms to the path 731456 in $g(T)$. □

Lemma 11. *For every digraph $D \in A_n$, there exists a unique labeled tree $T \in L'_n$, for which $g(T) = D$. i.e., g has an inverse.*

Proof. Let D be an arbitrary digraph in A_n. Below, we describe the procedure buildSubtree(r, C, T, D) that will be used to obtain the tree T, for which $g(T) = D$. The first argument, r, is a vertex in D and the second argument, C, is a cycle in D that contains r.

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buildSubtree(r, C, T, D):
    For each v ∈ C:
        if v ≠ r:
            add an edge (r, v) to E(T).
            if v is part of a cycle C_1 ≠ C:
                buildSubtree(v, C_1, T, D).
```

Initially, let T be an empty tree with $n+1$ vertices, i.e., let $V(T) = \{0, 1, \ldots, n\}$ and let $E(T) = \emptyset$. Take the vertex with label 1 in D. By Lemma 6, this vertex belongs to one or two simple oriented cycles. We consider each case separately.

Suppose that the vertex belongs to one such cycle and let C denotes this cycle. Then, run buildSubtree$(1, C, T, D)$ and add an edge $(0, 1)$ to $E(T)$. One can easily show that the resulting graph, T, is a tree in L'_n. First, T is connected since the execution of buildSubtree$(1, C, T, D)$ will reach every vertex of D and connect this vertex to an already reached vertex. In addition, if we have two different paths between two vertices u and v of T, then we will be able to find two simple oriented paths between u and v in D, which contradicts Lemma 6. Finally, $n \geq 2$ and the only vertex of T, which is not in the subtree with root 1, is the vertex 0. Thus, 2 is in that subtree and $T \in L'_n$.

Now, suppose that the vertex 1 belongs to two different cycles C_1 and C_2. Then, find the unique oriented simple path between 1 and 2 in D. This path has to have an edge in common with either C_1 or C_2, but not with both. Otherwise, we will have a contradiction with Corollary 7. Without loss of generality, let this be C_2. Execute buildSubtree$(0, C_1, T, D)$. The graph T, obtained at the end, will be a tree in L'_n (see Figure 6). □

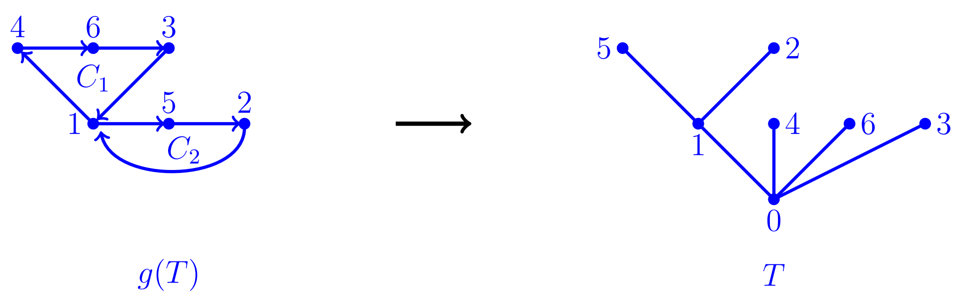

Figure 6: Example of the action of the inverse map g^{-1}.

Lemmas 10 and 11 imply that the map g is a bijection.

3 A bijection with parentheses arrangements

In this section, we give a second combinatorial proof of Theorem 3, via a bijection between the digraphs in A_n and a set of valid parentheses arrangements. Suppose that you have n pairs of opening and closing parenthesis, such that the two parenthesis in each pair are labeled with the numbers in $[n]$. A valid labeled parentheses arrangement is an ordering of these $2n$ parenthesis, such that we cannot have two interlaced pairs, e.g., $(_i(_j)_i)_j$ for some $i, j \in [n]$. A valid unlabeled parentheses arrangement is a sequence of unlabeled opening and closing parentheses that can be obtained by forgetting the labels of a valid labeled arrangement. One can easily check that a sequence of n opening and n closing parentheses is valid if and only if every prefix of the sequence has at least as many opening parentheses as closing parentheses. The number of valid arrangements of n unlabeled pairs of parentheses is C_n [3, Theorem 1.5.1]. Thus the number of such arrangements for labeled pairs is $n!C_n$.

To construct a bijection with the set of digraphs A_n, let us first note that one can assume that all vertices of a uni-Eulerian digraph have indegree and outdegree 2, provided that we allow digraphs with loops.

Lemma 12. *If B_n is the set of all uni-Eulerian digraphs on the vertex set $[n]$ (possibly with loops) with all vertices of indegree and outdegree 2, then $|B_n| = |A_n|$*

Proof. Given a digraph D in A_n, Corollary 5 implies that all vertices have outdegree 1 or 2 and the same indegree. Moreover, the single Eulerian tour of D passes exactly once through each vertex of outdegree 1. Hence, adding a loop to every vertex of outdegree 1 gives an element of B_n.

Conversely, given a digraph D' in B_n, deleting all loops gives an element of A_n. Indeed, the loopless digraph still has a unique Eulerian tour, which is just the tour for D' without the loops (the uniqueness follows because adding back loops must give an Eulerian tour for D'). These two maps are inverses of each other and thus give a bijection between A_n and B_n. □

Now, let B_n^* be the set of digraphs in B_n together with an identified edge. Since all digraphs in B_n have $2n$ edges, we have $|B_n^*| = 2n|B_n| = 2n|A_n|$. We will give a bijection between B_n^* and the set of valid arrangements of n labeled pairs of parentheses, which will show that $2n|A_n| = n!C_n$, that is, $|A_n| = \frac{1}{2}(n-1)!C_n$.

Theorem 13. *There exists a bijection between B_n^* and the set of valid arrangements of n labeled pairs of parentheses.*

Proof. [first part: Digraphs in $B_n^* \to$ valid parentheses arrangements].

Let D be a digraph in B_n^* with identified edge e. We define a parentheses arrangement $h(D)$ as follows:

Following the unique Eulerian tour of D, starting at e, open the i - th pair of parentheses when you pass through the vertex i for the first time and close the i - th pair of parentheses when you pass through the vertex i for the second time (see Figure 7 below) .

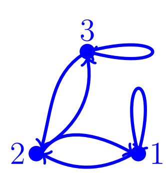

Figure 7: The digraph in B_3^* (with the edge $2 \to 1$ being identified) that yields the string $(_1)_1(_2(_3)_3)_2$

To show that the resulting string of parentheses is valid, we have to show that we cannot have two interlaced pairs of parentheses, e.g.,

$$\cdots (_i \cdots (_j \cdots)_i \cdots)_j \cdots .$$

In other words, the unique tour cannot have the form

$$i \xrightarrow{a} j \xrightarrow{b} i \xrightarrow{c} j \xrightarrow{d} i$$

for some walks a, b, c, d. But this is clearly impossible, because otherwise we would have a second Eulerian tour $i \xrightarrow{a} j \xrightarrow{d} i \xrightarrow{c} j \xrightarrow{b} i$.

[second part: Valid parentheses arrangements $\to$ Digraphs in B_n^*] Given a valid parentheses arrangement $w = (_x \cdots)_y$, we obtain a digraph $h^{-1}(w) \in B_n^*$ by putting an edge between the corresponding vertices of any pair of consecutive parentheses (from the first parenthesis to the second) and an edge from y to x. The identified edge is $y \to x$.

Clearly, every vertex in $h^{-1}(w)$ has indegree and outdegree 2 and that there exists an Eulerian tour T, given by the order of the parentheses' labels in w. Hence, we just have to show that T is the unique Eulerian tour of $h^{-1}(w)$. Let $i \in [n]$ and let

$$w = \cdots ?_\ell (_i (_j \cdots)_i ?_k \cdots ,$$

where ? represents either a closing or an opening parenthesis (if $(_i$ and $)_i$ are consecutive we let $j = i$, if $(_i$ is the first parenthesis of w we let ℓ be the label of the last one and if $)_i$ is the last parenthesis, we let k be the label of the first one). We have to show that if an Eulerian tour enters the vertex i for the first time from ℓ, this tour must exit the vertex i towards j and not k. Indeed, if this is true for all i, then the Eulerian tour is entirely determined by its first edge. Thus, this tour and T are equal up to a cyclic shift. Suppose, for the sake of contradiction, that there exists an Eulerian tour T' of $h^{-1}(w)$, which exits i towards k, after entering i for the first time, through ℓ.

Note that, by the properties of valid parentheses arrangements, the two parentheses corresponding to any vertex $v \neq i$ are either both between $(_i$ and $)_i$ (then we will say that v is of type A) or both outside (type B). Clearly, all edges of the graph $h^{-1}(w)$ with initial vertex of type A (respectively B) have a final vertex either i or of type A (respectively B), so the only way to go from a vertex of type A to a vertex of type B is through i and vice-versa. Therefore, since k is of type B, we must re-enter i in T' through a vertex of type B, in order to access vertices of type A. The only way to do so, however, is through the edge $\ell \to i$, which was already used in T'. This is a contradiction, so the uniqueness of the Eulerian tour is proved. The two described maps h and h^{-1} are obviously inverses of each other, so the proof is complete. □

4 Acknowledgments

We are thankful to Jan Kynčel for the suggestions in the mathoverflow post related to the considered problem [4].

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Concerned with sequences A102693