New bounds for the Vertex Folkman number $F(2_7; 5)$

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Abstract

Vertex Folkman number is defined as $F_v(a_1, a_2, .., a_r; q) = min\{|V(G)| | G \overset{v}{\to} (a_1, a_2, .., a_r)$ and $cl(G) < q\}$. Here $G \overset{v}{\to} (a_1, a_2, ..., a_r)$ means that for every coloring of vertices of G in r-colors, there exists a monochromatic clique K_{a_i}, for some color $i \in \{1, 2.., r\}$. The focus of this work is on the particular Folkman number $F(2, 2, 2, 2, 2, 2, 2; 5)$, abbreviated as $F(2_7; 5)$. It is known that $F(2_7; 5) > 15$ [2]. In this paper we show that $F(2_7; 5) > 17$ with the help of computer - assisted graph generation methods. The cases $F(2_7; 5) = 16$ and $F(2_7; 5) = 17$ were refuted consecutively, which led to the new lower bound. The main technique that we use is the Extend procedure described in [7]. In addition, some theoretical observations are made, which reduce the amount of necessary computation. The work finishes with a simple proof that the number 22 is an upper bound for the considered Folkman number.

1 Introduction

Vertex Folkman numbers belong to a branch of Ramsey theory. John Folkman was an American mathematician, who proved in 1970 that for $q > max\{a_1, a_2, .., a_r\}$, the number $F_v(a_1, a_2, .., a_r; q)$ exists [1]. The latter is defined as $F_v(a_1, a_2, .., a_r; q) = min\{|V(G)| | G \overset{v}{\to} (a_1, a_2, .., a_r)$ and $cl(G) < q\}$. Here $G \overset{v}{\to} (a_1, a_2, ..., a_r)$ means that for every coloring of vertices of G in r-colors, there exists a monochromatic clique K_{a_i}, for some color $i \in \{1, 2.., r\}$. We use the standard denotations cl(G)- the maximum number of mutually adjacent vertices of G and $\chi(G)$, where if $\chi(G) = k$, then k is the minimal number, such that the vertices of G can be partitioned into k sets, with no edges in each of these sets (e.g. partition of G into k independent sets). All graphs that this paper considers are simple and undirected.

In case $a_1 = a_2 = .. = a_r = 2$, the condition $G \overset{v}{\to} (a_1, a_2, .., a_r)$ is equivalent to $\chi(G) \geq r + 1$. Indeed, the existence of monochromatic K_2 clique for some color $i \in \{1, 2.., r\}$ means that the vertices of G cannot be separated into r sets without the presence of an edge in some of the sets. For the case $q = r - 1$, the following f was proved [2]:

Theorem 1: Let $r \geq 4$ be an integer number. Thus

a) $F(2_r; r - 1) \geq r + 7$

b) $F(2_r; r - 1) = r + 7$, where $r \geq 8$.

If $r < 4$ then the the corresponding Folkman number obviously does not exists. Theorem 1 clarifies the cases where $r \geq 8$. For all of the four remaining values, $r = 4, 5, 6, 7$, the exact numbers were discovered. The result $F(2_5; 4) = 16$ by Lathrop and Radziszowski [7] was the last to be received.

The following theorem concerning the case $q = r - 2$ was also stated by Nenov in [2]. The proof uses Theorem 1 and two additional lemmas.

Theorem 2: Let $r \geq 5$ be an integer number. Thus

a) $F(2_r; r - 2) \geq r + 9$

b) $F(2_r; r - 2) = r + 9$, where $r \geq 11$.

Here, the Folkman numbers in the cases $r = 5, 6, 7, 8, 9, 10$ still remain unknown. In particular, if $r = 7$, according to Theorem 2, $F(2_7; 5) \geq 16$. In other words, there is no graph G with less than 16 vertices for which $cl(G) < 5$ and $\chi(G) > 7$. In this paper, we obtain the better lower bound for this number $F(2_7; 5) \geq 18$, together with an easily received upper bound 22. The proof of the lower bound is partially theoretical and partially computational. The hypotheses $F(2_7;5)=16$ and $F(2_7;5)=17$ were refuted.An important initial step in both cases is the separation of a 3-anti clique from a graph, which is a potential solution i.e. has the two listed properties. After that, some sets of edges are added, such that no 5-clique is forced and the latter is made in all possible ways. Considering the second, more complex hypothesis $F(2_7;5)=17$, the described approach is applied twice. Here we use the idea described by Coles and Radziszowski and by Lathrop and Radziszowski in [6] and [7], respectively. Some already implemented software tools were used as well, namely Geng[9], Cliquer[10] and Smallk[11]. Before we get to the details of our work, we will mention one interpretation of the number $F(2_7;5)$, in order to shine some light on its nature. This simple interpretation works for all the other Folkman numbers after trivial modifications:

Interpretation: The Folkman number $F(2_7;5)$ is the smallest possible number of people in a group, such that among any 5 of them, at least 2 do not know each other, but on the other hand, if you decide to distribute them in 7 rooms, you will have at least two acquitances in some of the rooms and the latter is true for any distribution of the people over the 7 rooms.

2 Eliminating the case $F(2_7;5)=16$

First, we will describe the exact steps in the process of refuting the case $F(2_7;5)=16$. Let us assume the opposite i.e. there exists a graph with 16 vertices, with chromatic number more than 7 and without 5-clique as subgraph. Let us denote one such graph with G_0. It is known that the Ramsey number $R(5,3)=14<16$, then G_0 must contain a 3-anti-clique, because it doesn't contain a 5-clique. Let us denote one 3-anti-clique in G_0 with (a_1,a_2,a_3) and to separate it. The remaining graph

i must have 13 vertices

ii must be without 5- clique as a subgraph

iii must have chromatic number more than 6

The last constraint is due to the fact that after adding an independent set to some graph G_1, together with some edges in order to get graph G, the chromatic number of G can increase with at most one. Indeed, if $\chi(G_1)=k$, then a new color- $(k+1)$ can be assigned to each vertex of the independent set added to G_1. Thus, $\chi(G)\leq k+1$

In order to find the possibilities for the remaining graph, we will need the following theorem, which was proved in [3].

Theorem 3: Let G be a graph such that $\chi(G)-cl(G)\geq 3$ and $|V(G)|\geq \chi(G)+6$. Then $\chi(G)\geq 7$ and $G=K_{\chi(G)-7}+Q$ or $\chi(G)\geq 9$ and $G=K_{\chi(G)-9}+C_5+C_5+C_5$, where Q is the graph whose complementary graph is given in Fig.1. and C_5 is the cycle with 5 vertices.

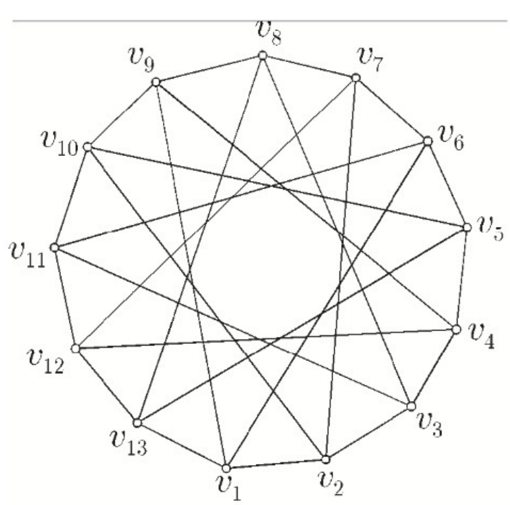

Figure 1: The complementary graph of Q

Here, the following definition of a sum of undirected graphs $G_1(V_1, E_1), G_2(V_2, E_2), ..G_k(V_k, E_k)$ is used:

$$G_1 + G_2 + .. + G_k = G(V, E).$$
$$V = V_1 \cup V_2 \cup .. \cup V_k,$$
$$E = E_1 \cup E_2 \cup .. \cup E_k \cup (\bigcup_{i \neq j} \bigcup_{\substack{u \in V_i, \\ v \in V_j}} (u, v));$$

From the first part of the latter theorem we deduce that there exists only one graph with 13 vertices, which has the described properties (ii) and (iii).This graph is Q (for simplicity, the complementary graph of Q is depicted, see Fig.1). The chromatic number of Q is exactly 7, which can easily be inferred from the theorem.

Now, we can apply the extend technique used in [7]. We need to investigate all the possible ways in which edges between the graph Q and the separated anti-clique (a_1, a_2, a_3) can be added. Therefore, for each of the three vertices a_i of the anti-clique, we will choose one set of vertices V_i of Q, with which it will be connected(see Fig.2). If any V_i contains 4-clique, then a 5-clique will be formed, because a_i will be connected with all the vertices of this 4-clique. Thus, we need to consider only those sets V_i which do not have as a subset some of the 4-cliques of Q. Moreover, we will bound ourselves to only those sets which are maximal by inclusion with this property. This is clarified below.

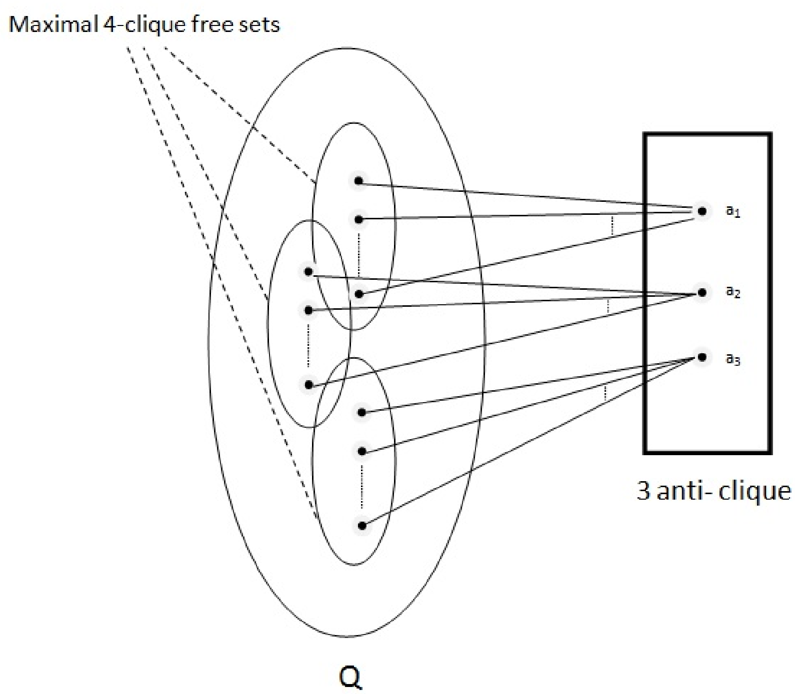

Figure 2: Extend technique applied to graph Q

Let us denote the set of all maximal 4-clique free subsets of Q with M. The graph received from Q, after adding the 3-anti-clique (a_1, a_2, a_3) and all edges from a_i to the set $V_i, i = 1, 2, 3$ will be denoted with $Q \oplus (V_1, V_2, V_2)$. For every possible triple of sets $<V_1, V_2, V_3>$ we will construct the corresponding graph $Q \oplus (V_1, V_2, V_3)$ and then test whether the chromatic number of this graph is 8 or 7 (same as the chromatic number of Q). Chromatic number testing was done with the help of the program Smalk, by J.Culberson [11].We are interested in the cases of increasing. If this happens, then for every 7-coloring of Q without monochromatic edge, at least one of the sets V_1, V_2, V_3 contains vertices of all of the 7 colors in this coloring. Obviously, if this condition is fulfilled for some triple $< V_1, V_2, V_3 >$, then it will be fulfilled also for every triple $< V_1', V_2', V_3' >$ such that for all i=1,2,3 $V_i \subseteq V_i'$ and V_i' is a 4-clique free subset of vertices of Q. This means that if we consider only the triples of maximal(in sense of inclusion), 4-clique free subsets and if no graph with chromatic number 8 is found, then this will be sufficient for us to conclude that this is true even if we try with triples of non-maximal sets.

The conducted calculations showed that among all of the constructed graphs, there is no graph with chromatic number 8 . 39 different 4-cliques were found in the graph Q. The count of the found maximal 4-clique free subsets of Q was 117, which makes 287391 triples of sets($\binom{117}{3} + 2 \times \binom{117}{2} + \binom{117}{1}$). This number is not very big and usually less than a day would be enough for an average PC to complete the work i.e. to construct the graph corresponding to each triple and to perform the chromatic number test. However, we have used the two additional facts described below, in order to reduce the amount of computation needed:

Fact 1: The graph $G_1 = Q \oplus (V_1, V_2, V_2)$ has chromatic number less than or equal to that of the graph $G_2 = Q \oplus (V_1, V_2, V_3)$, for every maximal 4-clique free subset of Q V_3, which is different from V_2. This allows us to consider only the triples of different sets in M.

Proof: Let us assume the opposite- $k_2 = \chi(G_2) < \chi(G_1) = k_1$. Thus, there exists a coloring of G_2 in k_2 colors. If γ is one such coloring, then after changing the color of a_3 in γ to be the same as that of a_2, we receive a new coloring γ_1 with k_2 colors for the graph G_1. Therefore, $k_2 \geq k_1$, which contradicts our assumption that $k_2 < k_1$.

The symmetry of graph Q can also be used substantially. If the set of vertices of Q is denoted with $\{0, 1, 2.., 12\}$, then we can define the relation $\overset{k}{\backsim}$ over sets of vertices of Q as follows: $U \overset{k}{\backsim} V$ if $U = \{x_1, x_2, ..., x_l\}$ and $V = \{(x_1 + k) \bmod 13, (x_2 + k) \bmod 13, ..., (x_l + k) \bmod 13\}$.

Fact 2: If $V_1 \overset{k}{\backsim} V_1'$, $V_2 \overset{k}{\backsim} V_2'$ and $V_3 \overset{k}{\backsim} V_3'$. then the graph $Q \oplus (V_1, V_2, V_3)$ is isomorphic to the graph $Q \oplus (V_1', V_2', V_3')$,

This observation allows us to reduce the number of triples of elements of M that we will consider 13 times. In the end, the graphs we constructed were only 20010 ($\binom{117}{3}/13$). As expected, the hypothesis $F(2_7; 5) = 16$ was rejected.

3 Eliminating the case $F(2_7; 5) = 17$

The approach using separation of 3-anti clique is used here, as well. As we stated in the previous section $R(5, 3) = 14$. This allows us to separate a 3-anti clique even from the graph that remains after the first separation. More precisely, if a_1, a_2, a_3 is a 3- anti clique in a graph G_0, such that $cl(G_0) < 5$ and $\chi(G_0) \geq 8$, then the remaining graph $G_1 = G_0 \setminus a_1, a_2, a_3$ will have 14 vertices, which means that another 3-anti clique b_1, b_2, b_3 exists there. If we define $G_2 = G_1 \setminus b_1, b_2, b_3$, then the two conditions below have to be fulfilled:

1) $|V(G_1) = 14|$, $\chi(G_1) \geq 7$ and $cl(G_1) < 5$
2) $|V(G_2) = 11|$, $\chi(G_2) \geq 6$ and $cl(G_2) < 5$

We started with receiving all the graphs that can be assigned to G_2. After that, the well-known Extend technique was applied, in order to get all possibilities for the graph G_1. In the end, the second Extension technique was performed with each of the newly obtained G_1 candidates. Unfortunately, among the received graphs with 17 vertices, neither had chromatic number 8.

First, let us explain how we got all of the 11-vertex graphs satisfying condition 2). Let us denote one such graph with H_0. There are two options for the graph H_0. The first one is that it is vertex-critical graph i.e. if any of its vertices is removed, then the chromatic number of the received graph decreases(it can decrease with at most one). The second option for H_0 is that it has a proper subgraph with chromatic number greater or equal to 6. We will need the following theorem which was stated and proved for the first time in 1983 [4]:

Theorem 4: Let G be a graph such that $G \overset{v}{\rightarrow} (2_r)$, $cl(G) < r$ and $|V(G)| = F_v(2_r, r)$, where $r \geq 5$. Then $G = K_{r-5} + C_5 + C_5$.

In the special case r=5, Theorem 4 gives us that the graph $C_5 + C_5$ is the only graph with 10 vertices, with clique number less than 5 and chromatic number 6 and that a graph with these properties with less than 10 vertices does not exist. Therefore, the two options for H_0 are the following : H_0 is either vertex-critical or it contains $C_5 + C_5$ as a subgraph. In the second case, we manually enumerated 21 non-isomorphic 11-vertex graphs with clique number 4 that contain $C_5 + C_5$. Let us investigate in more details the case in which H_0 is vertex-critical. With the help of the program Geng [9], we are able to generate all non-isomorphic graphs with 11 vertices bearing in mind to test each one for the properties listed in 2). However, the number of all these graphs is extremely big. That is why we need to make use of the following:

Fact 3: Let us denote the degree of a vertex v in a particular graph with d(v). If H_0 is a vertex-critical graph with 11 vertices, $cl(H_0) \leq 4$ and $\chi(H_0) \geq 6$, then for every vertex v of H_0 , $5 \leq d(v) \leq 8$.

Definition: A graph G has the Sperner property if $\exists v_1, v_2 \in V_G : (v_1, v_2) \notin E_G$ and $adj(v_1) \subseteq adj(v_2)$, where $adj(v) = \{u \big| u \in V_G, (u, v) \in E_G\}$ is the set of the vertices of G adjacent to v.

Lemma 1: If a graph G has the Sperner property, it cannot be vertex-critical.

Proof: Let G be a graph having the Sperner property and let us assume that it is vertex-critical. In addition, let $v_1, v_2 \in V_G$ such that $(v_1, v_2) \notin E_G$ and $adj(v_1) \subseteq adj(v_2)$ and $\chi(G) = k$. Then we know that the graph $G \setminus \{v_1\}$ must have chromatic number $k - 1$. If we take one $k - 1$ coloring γ of this graph, then the coloring $\gamma'(x) = \begin{cases} \gamma(x) & \text{if } x \neq v_1 \\ \gamma(x_2) & \text{if } x = v_1 \end{cases}$ is a proper $(k - 1)$ coloring of G without monochromatic edge, which is a contradiction.

Lemma 2: If G is vertex-critical graph, then $\forall v \in V_G \ (d(v) \geq \chi(G) - 1)$

Proof: We assume the contrary. This means that $\exists v \in V_G$ such that $d(v) < \chi(G) - 1$. Let v be one such vertex and let us consider the graph $G' = G \setminus v$. $\chi(G')$ must be $\chi(G) - 1$. If we take one proper coloring of G' in $\chi(G) - 1$ colors, then evidently, for coloring the adjacent vertices of v are used at most $\chi(G) - 2$ colors. Therefore, by Dirichlet's principal there exists a color which can be assigned to the vertex v, without creating a monochromatic edge. In result, we receive a proper coloring with $\chi - 1$ colors, which is a contradiction with minimality of $\chi(G)$.

Proof of Fact 3: First, let us suppose that there exists a vertex v, such that d(v)=10. Then the remaining graph $H_0 \setminus \{v\}$ must have chromatic number 5 and clique number less than 4. However, it has been known for a long time that F(2,2,2,2;4)=11 >10, so it is not possible for such a vertex to exists. In fact, this result was first published in 1984(see [5]). Now,we will see that if for some vertex $v \in V_{H_0}$, d(v)=9, then H_0 has the Sperner property. After that, the contradiction follows directly from Lemma 1. Indeed, if d(v)=9, for some $v \in V_{H_0}$ then we can take the only vertex u of H_0, that is not adjacent to v. Apparently, $adj(u) \subseteq adj(v)$ and $(u, v) \notin E_{H_0}$, which shows that H_0 has the Sperner property. Finally, Lemma 2 gives us the needed lower bound $d(v) \geq 5$, because $\chi(H_0) \geq 6$ and then $\chi(H_0) - 1 \geq 5$.

Fact 3 helped us a lot in finding all vertex-critical graphs with 11 vertices, which fulfill condition 2). After setting every vertex power between 5 and 8 inclusively, the number of graphs that were generated was smaller than 15 millions. For each of them we tested the chromatic and the clique number constraints using the programs Smallk and Cliquer [10].117 non-isomorphic graphs with chromatic number 6 and clique number 4 were received. As a result, the total count of those graphs became 138, together with the 21 graphs containing $C_5 + C_5$. Thus, all the possibilities for the graph G_2 were received.

The extension procedure with the 3-anti clique b_1, b_2, b_3 was executed for all of these 138 graphs. The number of triples of maximal sets without 4- cliques was comparatively small for each of them. Because of that, the computations here were not very time-consuming. We needed to filter only these graphs among the generated by the extension, which have chromatic number 7 (see condition 1)). Once again, the program Smallk came into use. Only 6 non-isomorphic graphs satisfied this condition. Here, the interesting result was that we got all of them by extending the same 11-vertex graph. The others 137 graphs did not give any 7-chromatic 14 vertex graphs after their extension. This unique 11-vertex graph is depicted in Figure 3. Explaining why this graph is so special remains the subject of further research.

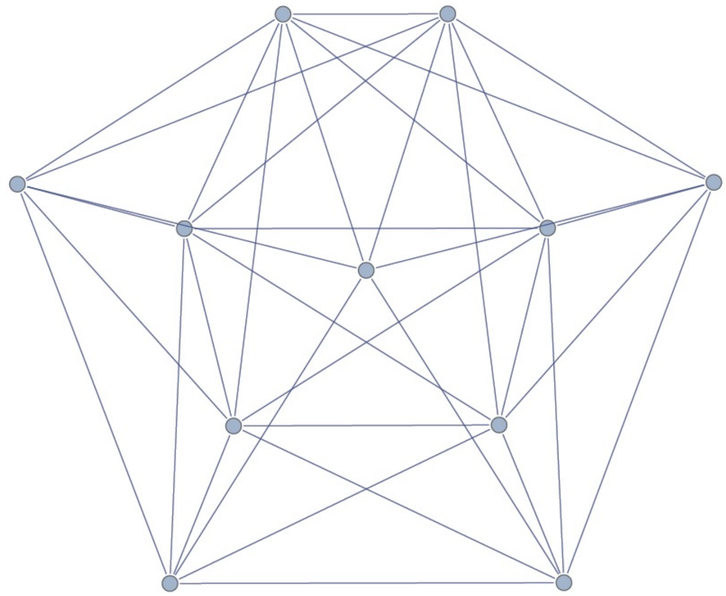

Figure 3: The unique 11-vertex graph G_2, which is contained in all of the graphs G_1, $|V_{G_1} = 14|$, $\chi(G_1) = 7, cl(G_1) < 5$

In the end, we executed the second extension procedure, with the 3-anti clique a_1, a_2, a_3 for each of the sixth 14- vertex graphs, which we found. This time, the computations were harder, because each of these graphs has more than 120 maximal, free of 4-cliques different sets of vertices. No graph with chromatic number 8 was found among all of the generated 17-vertex graphs in this step. We implemented our computer program in a very meticulous way, because, evidently, even one programming error would be fatal to the truthfulness of the results. This allows us to conclude that $F(2_7; 5)$ is at least 18.

4 Possible approaches to the hypotheses $F(2_7; 5) = 18$

In this case one can try to use the fact that the Ramsey number R(4,4)=18. If a 4-anti clique exists, we can exclude it and we must receive one of the 14-vertex graphs, that we have already considered. However, if a 4-anti clique is not present, the existence of a 4-clique in this 18-vertecs critical graph would not help a lot.

Another heuristic approach exists, which can be applied. An additional vertex, connected to each of the three vertex in the 3-anti clique can be added. This vertex can be connected with a subset of the remaining 14-vertex graph, not containing 3-clique(see Figure 4). With a powerful computer, such an approach can leads to the desired result, namely finding a graph with chromatic number 8 and clique number less than 5.

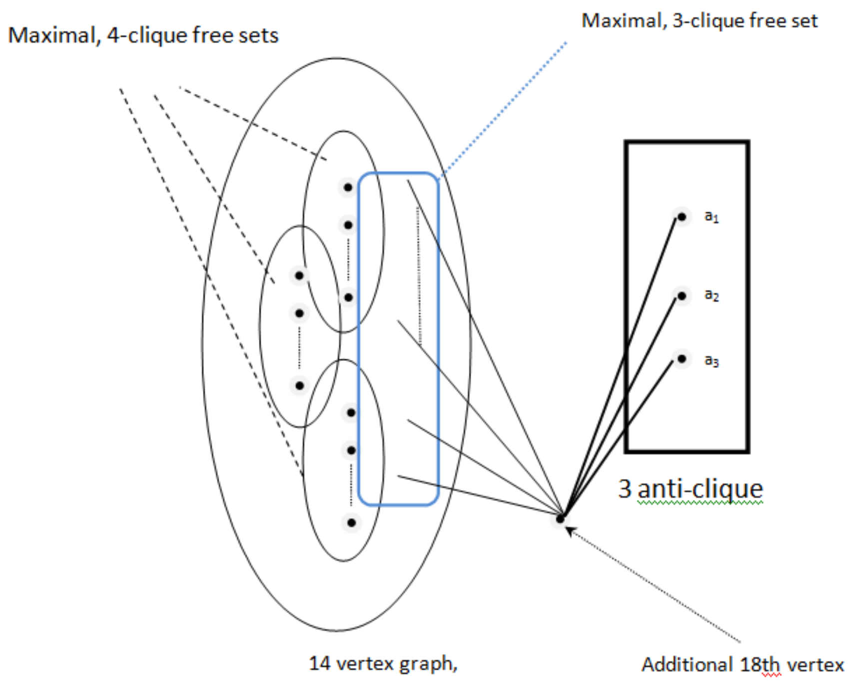

Figure 4

5 An upper bound

The Ramsey number R(5,4) is equal to 25([8]). Thus, for n<25 there always exists a graph G with n vertices, having cl(G)=4 and independence number $\alpha(G) = 3$. Let us take n=22. By Dirichlet's principal, it is not possible for such a graph with 22 vertices and $\alpha(G) = 3$ to have a chromatic number less than 8 (3.7=21<22). Therefore, 22 is an upper bound and it is the best that we can obtain from this observation.

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