In search of combinatorial proof

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Abstract

Some combinatorial identities were considered along with their bijective proofs. Focusing on the particular approaches used in these proofs, we derive a list of simple rules and associations that can guide us in the process of searching for bijective proofs in general. After reviewing more identities, this list could be extended and refined.

When proving a mathematical identity, we can use two main types of approaches- algebraic-analytical and purely combinatorial. If the former approach is used, we view the two sides of the identity entirely as algebraic objects, aiming to convert one of the sides into the other.On the other hand, if the latter approach is used, we construct a bijection between two sets, with cardinality equal to the left and the right side expressions, respectively. Let us illustrate the difference with the following

Problem 1: Prove the identity:

$$\binom{n}{r}\binom{2n}{n} = \binom{n+r}{r}\binom{2n}{n-r} \tag{1}$$

Proof:
Using the equality $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we transform (1) to :

$$\frac{n!}{r!(n-r)!}.\frac{(2n)!}{n!n!} = \frac{(n+r)!}{r!n!}\frac{(2n)!}{(n-r)!(n+r)!}, \tag{2}$$

which truthfulness can be easily seen after making the reductions.

Let us think about the problem in a different way. The left side of (1) can be associated with the number of ways by which we can choose n uncolored balls from a total of $2n$ uncolored balls and then r balls from the chosen n that will be painted with 2 colors. The remaining $n-r$ balls will be painted with only one color. The selections can be carried out in a different order as well - first,we can choose the $n-r$ balls, which will be unicolor (in $\binom{2n}{n-r}$ ways) and then we can choose the r balls that will be bicolored from the remaining $n+r$ in $\binom{n+r}{r}$ ways. Ultimately, we receive that the left and right sides of (1) are two expressions depicting the same thing in two different ways which confirms that (1) is an identity. Here, unlike with the first approach, we don't need to know any additional facts about binomial coefficients. The only thing we do is establish some parallels, though creatively constructed. This is the gist of the combinatorial approach. In some cases these types of solutions can be seen much more easily than the algebraic approach solutions. Furthermore, they are more intuitive and better reveal the meaning of the considered identity.

However, quite often a combinatorial solution cannot be seen as easily as in the previous problem. Therefore, it is a good idea to explore the process of searching for such a solution. It is also important to know what associations should be triggered by frequently occurring expressions. Once we have a set of useful correspondences, we only need to specify the details in the needed bijection. At the very least, we will get some ideas, which can be developed until a combinatorial proof is reached. Let us list some of these useful correspondences:

(i) binomial coefficient - $\binom{n}{k} \rightsquigarrow$ the number of ways to choose k elements from n;
(ii) Sum - $\sum_{i=y}^{x} \rightsquigarrow$ partition of a set on $(x-y)$ number of equivallence classes(relative to some equivallence relation);
(iii) Exponent - $a^b \rightsquigarrow$ The number of functions $f : \mathbb{A} \to \mathbb{B}, |\mathbb{A}| = a, |\mathbb{B}| = b$, i.e. choosing 1 out of a things, b number of times;
(iii') Power of 2 - $2^x \rightsquigarrow$ The number of the different subsets of a set with cardinality x. (iv) sum of terms with alternatively changing signs - $\sum(-1)^i ... \rightsquigarrow$ Inclusion-exclusion principle;
By using rules (i - iii), we can simplify the combinatorial solution searching process. This is demonstrated

by the example below.

Problem 2. Prove that for the natural numbers n, k the identity below is fulfilled :

$$\binom{2n}{k} = \sum_{j=\lfloor \frac{k+1}{2} \rfloor}^{k} \binom{n}{j}\binom{j}{k-j} 2^{2j-k} (k \leq 2n) \tag{3}$$

Proof:

Let us name "κ-choice" every selection of κ elements out of the initially given 2n (base elements).
On the right side of the equation we have a sum of about $\frac{k}{2}$ addend, which depends on k. Thus, it will be appropriate to try to find a partition of the aggregation of all k-choices(the expression on the left side), such that the number of equivallence classes, relative to this partition would be approximately the half of k. Moreover, we have $2n$ base elements , which we can choose from and the sum contains $\binom{n}{j}$ - selecting out of n objects(half of $2n$). These observations lead us to the idea of looking at the $2n$ elements as if they were situated in 2 rows(n columns, numbered from 1 to n) and counting the numbers of columns with at least one choosed element, for a fixed k-choice. If at least one element in a column is chosen, we will call it "not empty" and if both elements in the column are chosen, we call it "full". The maximal number of "not-empty" columns is k, provided that no column is full, while the minimal number is $\lfloor \frac{k+1}{2} \rfloor$ - when as many as possible of the "not empty" columns are "full".
Now we are really close to the idea of considering a partition on the bases of "number of non-empty columns". In this case the interpretation of $\binom{n}{j}$ will be- number of ways to choose a set of particular "non-empty" columns.

It remains to clarify the meaning of the expression $\binom{j}{k-j}2^{2j-k}$ in the right-hand side of the sum. The first multiplier gives the number of ways in which we can select $(k-j)$ columns out of the already choosen j columns. How could $k-j$ columns possiblly be different from the other "non-empty" columns? It is easy to see that the number of "full" columns out of all "non-empty" columns is exactly that much. Therefore, as we multiply by $\binom{j}{k-j}$, we fix the positions of "the full" columns. Let us make one last trivial observation, namely that if there is 1 chosen element in a column, then this element can be either in the first or the second row. However, the remaining $j-(k-j) = 2j-k$ columns are neither empty, nor full and are therefore exactly of this type. Thus, in order to fully determine our k-choice, we need to multiply by 2^{2j-k}.

In summary, identity (3) can be deduced directly through summing the number of elements in the different equivallence classes on relation $R \doteqdot \{(S_1, S_2) \mid S_1, S_2$ are k-choices and have equal number of "non-empty" columns $\}$. R is evidently an equivallence relation.

We continue with an example of an application of association (iv) in

Problem 3. Prove that

$$\sum_{k=1}^{n-1} (-1)^{k-1} \binom{n}{k} (n-k)^{n-2} = n^{n-2} \tag{4}$$

Proof:

Here, we have one big sum again, but it has alternatively changing term signs, thus we would not search for partition. Naturally, we start with the simpler side of the equality, which in our case is the right side. The powers, according to (iii), remind us to consider a rectangular table with n-2 rows and n columns(numbered from 1 to n). n^{n-2} is the number of ways in which we can choose one entry in every row of the table. We will call such choosing - "table choosing". Correspondence (iv) leads us to the assumptuion that the left hand side is received as a result of the inclusion-exclusion principle applied over sets of table choosings with union - the set of all table choosings(that has cardinality n^{n-2}). Our goal will be to make a reasonable suggestion about the entity of these sets of choosings. We know that the κ-th consecutive addend in the sum has to be equal to the sum of cardinalities of all the intersections of exactly κ from the wanted sets. It has the form $\binom{n}{k}(n-k)^{n-2}$. Apparently, by chosing k out of n elements here, we can connect the selection of the k sets, which we intersects, i.e. every set has a corresponding column in the table. On the other hand, $n-k$ is the number of unchosen sets(columns). Knowing also the well-known interpretation of $(n-k)^{n-2}$, we reach the most natural idea about the nature of the sets, on which we apply the principle. More precisely:
$A_i = \{u | u$ -table choosing and u has no chosen entries in column number i$\}, \forall i = 1, 2, ...n$.

It is not difficult to see that $|\bigcap_{j=i_1,..,i_k} A_j| = (n-k)^{n-2}$, thus we can conclude with sufficient certainty that equality (4) is true, as a result of the application of the principle over the sets $A_i, i = 1, ..n$. The latter problem is a classical example of the usefulness of combinatorial proofs.

References:

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