Homework 1 (01:640:356 THEORY OF NUMBERS)

Spring 2023

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1. **Lucas numbers**

   The Lucas numbers L_n are defined by

   $$L_n = L_{n-1} + L_{n-2},$$

   where $L_1 = 1$ and $L_2 = 3$. They satisfy the same recurrence as the Fibonacci numbers f_n, but the initial values are different. Show by induction that

   a) $L_n = f_{n-1} + f_{n+1}$.

   b) $f_{2n} = f_n L_n$.

2. **Conversion**

   Convert $453_{(9)}$ from base 9 to base 3.

3. **Cantor's expansions**

   Find the Cantor's expansions for the numbers 355 and 1214.

4. **Primes' infinitude from factorials**

   Show that there are infinitely many primes by looking at the integers $n! - 1$, where n is a positive integer.

5. **Primes ending with 1s**

   Show that for every positive integer n, there is a prime whose decimal expansion ends with at least n 1s.

6. **Binomial coefficient**

   Let $n > 3$ be a positive integer and let p be a prime, such that $\frac{2n}{3} < p \leq n$. Show that p does not divide the binomial coefficient $\binom{2n}{n}$.

7. **Cool primes**

   Let C be the set of all positive integers of the form $4k+1$, for a nonnegative integer. An element $c \neq 1 \in C$ is called a *cool* prime, if the only way to write it as a product of two integers in C is $c = c \cdot 1$. Show that every element of C greater than 1 can be factored into cool primes. Show that 693 has two different factorizations into cool primes.

8. **Chinese Reminder Theorem**

   Find the smallest multiple of 10, which has remainder 3 when divided by 7, and remainder 2 when divided by 3.

9. **Special number**

   An integer n is called *special* if, whenever a prime p divides n, p^2 divides n. Show that every special number can be written as a product of a perfect square and a perfect cube.

10. **Powers**

    Show that $(a^m - 1, a^n - 1) = a^{(m,n)} - 1$, whenever a, m and n are positive integers and $a > 1$.

11. **Gcd and linear combinations**

    Find and expression of the greatest common divisor of 1234 and 981 as a linear combination of these two numbers.

12. **Bertrand's postulate**

    Use Bertrand's postulate that for every integer $u > 1$, we have a prime between u and $2u$, to show that $\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{n+m}$ is <u>not</u> an integer when n and m are positive integers.

    Hint: Consider the cases $m < n$ and $m \geq n$ separately.