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The Universities of Berlin and Their Role In Developing Mathematics

Mathematics is a foundational skill that is utilized in almost all other fields of work. However, the mathematics we know of today is a result of the work of countless individuals throughout history, all building off of the work of their predecessors and fellow mathematicians. The field of mathematics consists of many topics, each of which have their own histories of development, originating from people all over the world during various time periods. However, during certain time periods some regions are recognized as major contributors to the development of various mathematical topics. When discussing the development of the math we know of today, it is hard not to mention Germany and the several findings within this country from the 17th century onward. Germany's history consists of strong math periods with several notable mathematicians, especially originating from the universities of Berlin. This city is rich in its contributions to the field of mathematics, which later influenced the work of several other future mathematicians. Berlin influenced the development of several fields of mathematics into what we know today so it is important to explore why these cities prospered educationally and see some of the foundational findings that would change the notation and perspective of future mathematics.

Berlin was first recognized as a small town around 1237 located in the state later known to be Prussia, founded by hunters and merchants, but it wasn't until the end of the 17th century when Berlin started its journey into the world of mathematics. During the late 1600s, Berlin was recovering from the Thirty Years War (1618 - 1648) which caused Berlin's population to roughly half from 12,000 to 7,500 citizens due to starvation and expensive markets. However it was this period of recovery during the 1650s up to the early 1700s that allowed Berlin to reconstruct itself and its offerings. First, The Great Elector of Brandenburg Fredrick William focused on rebuilding the state's towns and cities (including Berlin), building a strong army, and reorganizing finances, all of which together created a strong foundation that allowed Berlin to rise in popularity. During this time, various other locations started to construct artistic, scientific, and mathematical institutions or societies. A few examples of this include the Royal Society of London which was made in 1660 and the French Academy (or Paris Academy) of Sciences in 1666. Therefore, his successor Fredrick III established an Academy of the Arts in 1696.

Frederick III also wanted to create a new kingdom (state), Prussia, of which he would be king so he could make cultural changes. One of the tasks to do so required him to create a new regional Prussian calendar. Gottfried Wilhelm von Leibniz, a prominent mathematician, historian and diplomat of the time, advised Fredrick III to establish "a society to undertake the scientific and technical work involved in producing accurate almanacs and calendars" (Craig). Leibniz' main interest was to revitalize German education like the neighboring regions by creating an institution to promote scientific discussion and communication. Frederick III agreed to build such an institution once Leibniz highlighted that sales of the calendars would self-finance the institution so Fredrick III would not have to alter his budgets and he would gain prestige. Therefore, the Berlin-Brandenburg Society of Scientists was created in 1700, as well as Prussia during early 1701. The society went by many names throughout history, including the Prussian Academy, the Berlin Academy of Science, Royal Academy of Sciences and many more.

Berlin was named the royal city of residence, and Leibniz was named as one of the founders and starting president of the Berlin Academy. These developments amongst others made Berlin very attractive to others, especially mathematicians seeing that it had the support of Leibniz. Between 1670 to 1712, the population grew from 12,000 to 61,000 and continued to grow.

By the early 1700s, Gottfried Wilhelm von Leibniz was a recognizable name to many. He lived between 1646 - 1716, mostly in Germany. He was born during the end of the Thirty Years War, and went to was largely self taught by his father despite attending public school. He later attended the University of Leipzig for Law, which helped him come into contact with famous scientists and philosophers such as Galileo, Francis Bacon, Thomas Hobbes, and René Descartes. It was through these figures Leibniz grew an appreciation toward education and wanted to help society learn. In 1666 he wrote De Arte Combinatoria ("On the Art of Combination"), in which he formulated a model that is the theoretical ancestor of some modern computers: all reasoning, all discovery, verbal or not, is reducible to an ordered combination of elements, such as numbers, words, sounds, or colors (Britannica).

Leibniz spent the following years on philosophical work but in 1673 he started to pursue his mathematical and scientific studies again. He created a "calculating machine" also known as the Step Reckoner, which was one of the first mechanical gear based calculators able to do addition/subtraction and multiplication/division with 8-digit numbers. On top of this, in 1675, Leibniz revealed his work on integral calculus, which is arguably his most well known work. Up until this point, calculus worked within limits but Leibniz introduced how to work with infinitesimal quantities and introduced the notation used today. Therefore, *infinitesimal* calculus allowed for precise analysis of functions within continuous domains. The other major contributor toward calculus at the time was Sir Isaac Newton, and he used a small vertical bar above a variable to indicate integration, or placed the variable inside a box. But Leibniz adapted the integral symbol, ∫, from the letter ʃ (long s), standing for summa (written as Summa; Latin for "sum" or "total") (SciHi). Furthermore, Leibniz described what we now know as the product rule (shown on the right) in calculus, and also discovered $d(x^n) = nx^{n-1}dx$. By the 1680s up til the early 1700s, Leibniz focused on a new science of Dynamics, which was yet to be founded, and applying his physics research on inventions such as windmills, lamps, clocks. All of this together popularized Leibniz' name in the mathematical world of the time.

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Membership into the Berlin Academy was made by royal appointment and members received special privileges. The Academy offered lectures regarding mathematics, physical sciences which would apply some of the mathematics taught, and other humanities based lectures. These lectures were given by members or invited speakers. During the 1700s, as president of the Berlin Academy, Leibniz would teach his past discoveries and philosophies. His past work on calculus was taught to individuals, along with findings in geometry, algebra, linear equation systems, and his refined binary numerical system. Although Leibniz himself did not conduct many lectures. Despite being president he also retained his previous position of work in Hancover. This made it so that Leibniz spent limited time in Berlin. Between 1700-1711, Leibniz acted as president for a total of ten terms (about three years).

Therefore, the Academy started publishing an annual journal that would describe the research of the Academy and the organization of the institution. The first volume, *Miscellanea Berolinensia*, was released in 1710 and consisted of 61 publications, 12 of which were written by Leibniz. Of these 12, three publications related to mathematics: His index notation which greatly helped his research in determinant theory, analogies between the powers of a polynomial and the differentials of a product (which would describe his work on infinitesimal calculus), and lastly an application of differential calculus to a tangent problem (Some of these topics will be elaborated on below). On top of these journals, members such as Leibniz often wrote manuscripts or articles describing their work which was useful for future development of these ideas. Leibniz' organization of linear equations into a matrix paved the way for Carl Friedrich Gauss and his work on linear algebra later on.

Other examples of Leibniz' mathematical work described either through the annual journals of the Academy or manuscripts written are the Leibniz formulas for determinants, for π, and the Leibniz Integral Rule. The Leibniz formula for determinants expresses the determinant of a square matrix in terms of permutations of the matrix elements. The formula reads

$$\det(A) = \sum_{\tau \in S_n} \operatorname{sgn}(\tau) \prod_{i=1}^{n} a_{i,\,\tau(i)} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),\,i}$$

where sgn is the sign function of permutations in the permutation group S_n which returns +1 and -1 for even and odd permutations respectively. This equation on its own is not efficient to find the determinant of a square matrix anymore

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because of more efficient methods such as LU decomposition, however the equation is still used by physicists and in proofs to show properties of determinants. The Leibniz formula for π is a special case of the arctan Taylor Series which reads

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

turns into

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$

The proof of the equation above is shown below.

$$\begin{aligned}\frac{\pi}{4} &= \arctan(1) \\ &= \int_0^1 \frac{1}{1+x^2}\,dx \\ &= \int_0^1 \left(\sum_{k=0}^{n}(-1)^k x^{2k} + \frac{(-1)^{n+1}\,x^{2n+2}}{1+x^2}\right)dx \\ &= \left(\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\right) + (-1)^{n+1}\left(\int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx\right).\end{aligned}$$

Considering only the integral in the last term, we have:

$$0 \le \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx \le \int_0^1 x^{2n+2}\,dx = \frac{1}{2n+3} \to 0 \text{ as } n \to \infty.$$

Therefore by the squeeze theorem and as n approaches infinity, only the Leibniz series is left,

$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

The Leibniz formula of π described through articles from the Academy of Berlin was an application of Leibniz' work on calculus regarding integrals. But it is also symbolic of Leibniz' mindset on education in terms of expanding upon the work of others and spreading awareness. Lastly the Leibniz Integral Rule is also known as the method to differentiate under the integral. This rule allows you to find the derivative of an integral without having to compute the integral itself, which is useful for integrals that do not have elementary antiderivatives. There are two applications of this rule. One of which is for a finite region (constant limits) and one is for a variable region (functional limits). The general equation reads as follows,

$$\frac{d}{dx}\left(\int_{\alpha(x)}^{\beta(x)} G(x,t)\,dt\right) = G\big(x,\beta(x)\big)\cdot\frac{d}{dx}\beta(x) - G\big(x,\alpha(x)\big)\cdot\frac{d}{dx}\alpha(x) + \int_{\alpha(x)}^{\beta(x)}\frac{\partial}{\partial x}G(x,t)\,dt$$

For example, let us consider the following function with a functional limit being x:

$$\int_0^x (x-t)^2\cdot u(t)\,dt$$

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We can see that characteristics below are true:

$$\alpha(x) = 0,\ \frac{d\alpha(x)}{dx} = 0$$

$$\beta(x) = x,\ \frac{d\beta(x)}{dx} = 1$$

$$G(x, \alpha) = (x - 0)^2 \cdot u(0) = x^2 \cdot u(0)$$

$$G(x, \beta) = (x - x)^2 \cdot u(x) = 0$$

$$\frac{\partial G}{\partial x} = 2(x - t) \cdot u(t)$$

Differentiating the function and replacing values in the Leibniz integral rule gives us the following:

$$\frac{d}{dx}\left(\int_0^x (x - t)^2 \cdot u(t)\, dt\right) = 0 \cdot 1 - (x^2 \cdot u(0)) \cdot 0 + \int_0^x 2(x - t) \cdot u(t) dt$$

Which is equivalent to

$$2\int_0^x (x - t) \cdot u(t) dt$$

.

Now for an example where there are constant limits. Given that the derivative of constants is zero, the general equation can be simplified to

$$I'(y) = \frac{d}{dy}\left(\int_a^b f(x, y)\, dx\right)$$

$$= \int_a^b \frac{\partial f(x, y)}{\partial y} dx$$

Let us consider the following function:

$$I = \int_0^1 \frac{x^k - 1}{\ln x} dx$$

.

Using the Leibniz Integral rule we can show

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$$\frac{dI(k)}{dk} = \int_0^1 \frac{\partial}{\partial k}\left(\frac{x^k - 1}{\ln x}\right)dx$$
$$= \int_0^1 \frac{x^k \ln x}{\ln x}dx$$
$$= \int_0^1 x^k dx$$
$$= \frac{1}{k+1}$$

$$I(k) = \ln(k+1) + C$$

And if we integrate both sides, we get the non derivative to the right.

Leibniz was the intellectual life of the Berlin Academy during its early years in terms of mathematics, greatly boosting its popularity. However, the profits generated by the calendars were taken up by Fredrick III (renamed to Fredrick I after becoming the first king of Prussia) and besides providing for maintenance costs, all other profits were used for reasons besides improving the Academy itself. Frederick's immediate successor treated the Academy in the same manner, so from 1712 onward, the Academy seemed to have reached a plateau in popularity and growth., especially as neighboring Academies in London and Paris continued to grow. It wasn't until 1740 when the Academy started to grow once again thanks to the leadership of Fredrick II. With the goal of opening to wider international influences, Frederick II introduced a new curriculum including new classes in experimental philosophy, speculative philosophy, mathematics and literature in 1746. A new administration was established and the funds from the almanacs were returned.

He also actively searched for new international scientists to add to the faculty list. He was able to convince the famous Swiss mathematician Leonhard Euler to serve as the Director of Mathematics in 1741 while also inviting the French mathematician Pierre Louis Moreau de Maupertuis to serve as Academy President. Other notable mathematicians of the time that joined the Berlin Academy include Johann III Bernoulli, from 1764 until his death in 1807, Johann Heinrich Lambert, from 1764 until his death in 1777, and Joseph Louis Lagrange, from 1766 until 1787. Lagrange, another major name in the mathematical world of the time, only agreed to join after Euler left in 1766, believing that he would be viewed second to Euler if he were to join the Academy at the same time. The last major change to the Berlin Academy was that international prizes were introduced and the court guaranteed salaries in excess of those offered in Paris and London. Frederick made French the official language and papers were published in French, and German, allowing the academy to compete with Paris and London (Craig).

Johann III Bernoulli spent his time at the Academy working on Astronomy and mathematics specifically in probability, recurring decimals and the theory of equations. His work *Leipzig Journal for Pure and Applied Mathematics* was published between 1776 and 1789 and was taken very seriously because he came from a very impressive bloodline full of famous mathematicians. He also spent a lot of time translating the works of other mathematicians into French for other members of the Academy. Johann Heinrich Lambert was the first to introduce hyperbolic functions into trigonometry and he also made conjectures that amount to form the properties of non-euclidean spaces (curved higher order spaces). He is also credited with the first proof showing that π is irrational in 1761. His proof first shows that the continued fraction expansion on the right holds true. He then proves that if

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots}}}}.$$

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x is non-zero and rational, then this expression must be irrational. Since tan(π/4) = 1, it follows that π/4 is irrational, and therefore π is also irrational. However Lambert's proof was not enough to convince Euler, and became a common talking point between the two of them after he joined the Berlin Academy.

Leonhard Euler was a Swiss mathematician who lived between the years of 1707 - 1783. Euler's father, Paul Euler, had worked with the famous mathematician Jacob Bernoulli earlier in his life so he was able to introduce Leonhard Euler into the world of mathematics, who was inspired and made it a habit to do mathematical related reading in his free time. He attended the University of Basel where he met the famous mathematician Johann Bernoulli and convinced him to conduct private lessons. It was this time that inspired Euler to switch careers and focus on mathematics later in his life after attempting to follow the steps of his father as a philosopher. Therefore, in 1726, he completed his mathematical studies at the University of Basel, while continuing lessons with Bernoulli where he studied popular works of the time. In 1727 he published an article on reciprocal trajectories and submitted an entry for the 1727 Grand Prize of the Paris Academy on the best arrangement of masts on a ship. Although Euler won second place, it was impressive for a young graduate and won him popularity, enough so that he was offered a teaching role at St Petersburg, which he accepted in November 1726. However, between the years 1727-1730, Euler spent time on the side as a medical lieutenant in the Navy as he slowly got promoted at St. Petersburg. By 1733, Euler served as the senior chair of mathematics. During this time Euler published many articles and his book *Mechanica* (1736-37), which extensively presented Newtonian dynamics in the form of mathematical analysis for the first time. By 1740, Euler was very reputable and was able to win the grand prize of the Paris Academy in 1738 and 1740. It was then he was offered a prestigious role at the Berlin Academy however Euler preferred his post at St. Petersburg. However, political conflicts helped change Euler's mind and in 1741 he joined the Berlin Academy for over 25 years.

Euler is known for his work in many fields of mathematics, including number theory, graph theory, logic, applied mathematics, real analysis, geometry, astronomy and more. He is argued to be the greatest mathematician of the 18th century, and is heavily praised by his peers. During his time at the Berlin Academy alone, he wrote 380 works of which 275 were published. He commented on the findings of his peers and engaged in discussion with many of them at the Berlin Academy. Euler is responsible for many of the notations that we see today, including the special characters π, for the ratio of the circumference of any circle to the diameter of that circle, e, for Euler's number, *i* for a complex variable, and f(x) notation. To highlight some of his greatest contributions, Euler popularized the use and development of a power series, which was originally created by Isaac Newton. Euler showed that the base of natural logarithms can be computed using an infinite series.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right).$$

Euler also created perhaps his most famous work, the Euler formula and the Euler Identity. The Euler formula reads

$$e^{ix} = \cos x + i \sin x,$$

Its significance is that it was an equation that was able to create a fundamental relationship between trigonometric functions and the complex exponential function. Euler, a frequent user of power series expansions, was able to prove this formula by relating the power series expansions of e, cos, and sin together. The proof goes as follows. It was already proven that the power series expansions of sin and cos, and e are

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$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

It is also known that the powers of *i* are the following

$$\begin{array}{llll} i^0 = 1, & i^1 = i, & i^2 = -1, & i^3 = -i, \\ i^4 = 1, & i^5 = i, & i^6 = -1, & i^7 = -i \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

With this in mind, we can say

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= \cos x + i \sin x, \end{aligned}$$

This equation had many applications in complex number theory and trigonometry. Soon, it was proven that

$$\begin{aligned} \cos x &= \operatorname{Re}(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2}, \\ \sin x &= \operatorname{Im}(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}. \end{aligned}$$

and

$$e^{i\pi} + 1 = 0$$

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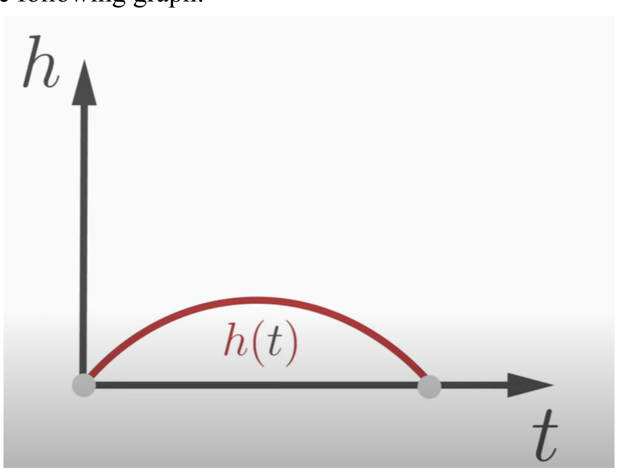
which is known as Euler's Identity. These discoveries combined were picked up by several mathematicians, physicians and others to be applied in many fields that involved the complex plane. For instance, Euler's Formula is used many times in Laplace transforms, which was made during the early 1800s by Pierre-Simon, marquis de Laplace. These Laplace transforms were then heavily used in classical mechanics and electrical circuit design later into the future.

Euler was the Berlin Academy's second major contributor after Leibniz, followed by Joseph Louis Lagrange. Lagrange joined the Berlin Academy during 1766, succeeding Euler as the director of mathematics. Lagrange is argued to be the second best mathematician of the 18th century, as he as well has lots of contributions in many areas of mathematics including Analytical mechanics, Calculus of variations, Celestial mechanics, Mathematical analysis, Number theory, Theory of equations, and more accompanied by several prizes and accolades. Most of his work during his Berlin years related to Astronomy and Physics, however in 1770, he presented his important work *Réflexions sur la résolution algébrique des équations* which made a fundamental investigation of why equations of degrees up to 4 could be solved by radicals. The paper was the first to consider the roots of an equation as abstract quantities rather than having numerical values. Alongside this, he also studied permutations of the roots. Although he does not compose permutations in his work, many consider it to be the first step in the development of group theory which would later be continued by Ruffini, Galois and Cauchy (mathsHistory).

During the 1750s, leading into the 1760s, Euler and Lagrange worked together to develop the Euler-Lagrange Equation, which is important in Calculus of variations. This branch of calculus focuses on using variations, which are small changes in functions or functionals (a function that takes a function as an input and outputs a real number) to find maxima and minima. The Euler-Lagrange Equation is useful in solving optimization problems where given a functional, you need to find the maxima and minima. The equation reads as

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0.$$

To understand the equation, we will work through a classical mechanics example. Consider a particle in a gravitational field, which is thrown vertically upwards. It travels straight up and down, reaching a starting and ending height of 0 meters and leaves at t_1 and arrives back on the ground at t_2. If we were to graph this behavior, we would see the following graph:



But if we wanted to measure the action value of this path (the scalar quantity describing how a physical system has changed over time, measured in Joule seconds), a unit that is important in mechanics and physics, then we would

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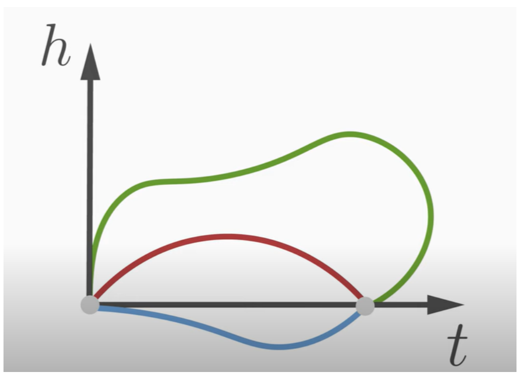
utilize a Lagrangian function. A Lagrangian function denoted by L, takes in three parameters: time, a coordinate function h(t), and the derivative of the coordinate function h'(t). By taking the integral of a Lagrangian function with these parameters between t_1 and t_2, we can compute the action value of the path described by h(t). The equation looks like the following, and note that it represents S[h] which is a functional.

$$S[h] = \int_{t_1}^{t_2} \mathrm{d}t \, L(t, h(t), \dot{h}(t))$$

where h(t) with a dot denotes the derivative of the function. However, why is it that the particle took the parabolic path denoted by h(t), instead of any one of the infinite other paths between the starting point and ending point of the time axis? That is, why not take the path above (green line) or the path below (blue line) of our h(t) parabolic path?



This is due to the fact that nature is extremal, which in simple terms is a property that says nature will take the path that produces either a minimum action value, maximum action value, or at a saddle point. There are ways to figure out which of these three nature will take, but the answer varies per problem. For our scenario, nature will take the path with minimum action. To avoid computing the action value of all possible paths, the Euler-Lagrange equation can be used. It is a tool to find optimizations, that is, given a functional such as a lagrangian function, it can find the minima and maxima. In most cases of classical mechanics, the Lagrangian function is found by the difference of the kinetic energy and the potential energy of a particle. These are listed below:

$$L(t, h, \dot{h}) = W_{\mathrm{kin}}(t, h, \dot{h}) - W_{\mathrm{pot}}(t, h, \dot{h})$$

$$\frac{1}{2} m \dot{h}^2 \qquad\qquad m g h$$

From here, we can follow the Euler-Lagrange equation. Note that here, h(t) is taking the place of q(t).

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$$L = \frac{1}{2} m \dot{h}^2 - m g h$$

$$\frac{\partial L}{\partial h} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{h}} = 0$$

After taking the partial derivative with respect to h, and h', and the time derivative, we arrive at

$$-m g - m \ddot{h} = 0$$

Which is equivalent to

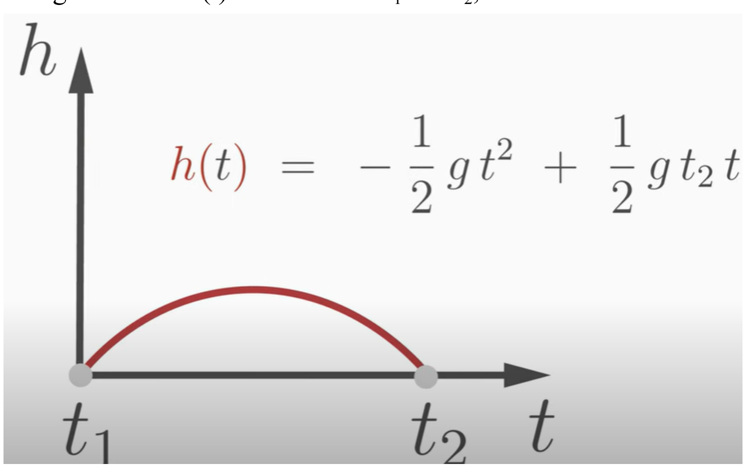
$$-g = \ddot{h}$$

If we take the integral of this equation twice, we arrive at

$$h(0) = 0 \qquad h(t_2) = 0$$

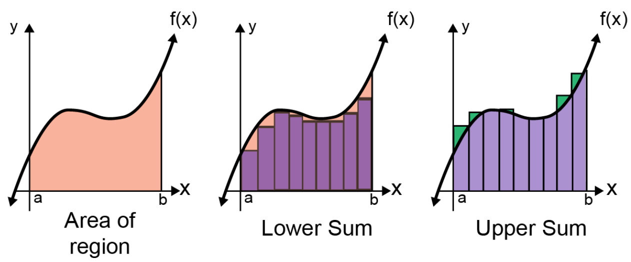
$$h(t) = -\frac{1}{2} g t^2 + C_1 t + C_2$$

Using the initial h(t) conditions at t_1 and t_2, we can find the C values and arrive at our answer for h(t),



Again, this h(t) was arrived at by the Euler-Lagrange equation, meaning this is the suggested path that nature will take when a particle is thrown vertically in a gravitational field. As we can see, the h(t) function is indeed parabolic as expected. This example showcases how the Euler-Lagrange equation is a useful tool in mechanics and physics to deal with optimizing functionals such as the Lagrangian function in order to fund coordinate functions with either a minimum or maximum action value. (Diagrams by Universaldenker Physics)

By 1809, the University of Berlin was constructed, and Berlin Academy started to have less of a role in the development of mathematics, although still an ongoing presence. The University of Berlin was formally named Humboldt University of Berlin, and it developed into the largest university in Germany. It enrolled more than 1,750 students by 1840 and became a leader in teaching and research. It had a slow growth due to conflicts with the French, a story that would be repeated in the forms of different conflicts with neighboring regions throughout the century, but the University of Berlin still gained popularity and prestige. Up to this day, the university has been associated with 57 nobel prize winners, ranging throughout multiple fields. During the 19th century, some of the most notable mathematicians/physicists to either attend or lecture at the University of Berlin includes Gotthold Eisenstein, Edmund Landau, John von Neumann, Ludwig Scheeffer and Albert Einstein. Gotthold Eisenstein was a German mathematician who took classes at the University of Berlin at 17 years old during the year 1840. By 1844, he created and presented his first work, on cubic forms in two variables, to the Berlin Academy. He published roughly 23 papers in Crelle's Journal, along with two problems which included two proofs of the law of quadratic reciprocity, cubic reciprocity and quartic reciprocity. By 1847, he began to teach at the University of Berlin. Bernhard Riemann, another growing German mathematician, attended some of his classes for roughly two years. Riemann is another popular mathematician who established Riemannian geometry, which played a crucial role in Albert Einstein's general theory of relativity. Riemann is also responsible for creating the Riemann sum, published some time around 1870, which is a topic all modern day college students learn in introductory calculus. The Riemann sum is an approximation method to approximate the area under a curve, or in other words, approximate the value of an integral. The Riemann sum takes a curve on a graph, and splits it up into several small vertical rectangles. The area of these rectangles are then added up to generate an approximation of the area under the curve. This leads to the formal equation shown below.



$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x_i .$$

By the end of the 19th century, mathematics was still prominent in Berlin but many other locations throughout the world had their own institutions and universities that caused the rate of growth in Berlin to decline. This paired with political issues of the time caused the development of mathematics to still progress, but at a slower rate in Berlin specifically. However, it is evident that Berlin played a crucial role in the history of Mathematics especially during the 18th and 19th century. Several notable mathematicians crossed paths with the city and published many works that would pave the path for others in the future. These findings changed the perspectives of many, causing more people to appreciate mathematics and its applications in other fields such as physics as more and more inventions were being made to benefit society. Mathematics is a collective effort, the work of many, and as it continues to develop in our modern day, more tools will be made available to create more impressive technologies.

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