01:640:437 HISTORY OF MATHEMATICS

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W9: The development of Probability and Statistics.

March 23, 2023

Ancient understanding of chance

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– Excavations from the pyramids in Egypt found *astralagi* (dice crafted from animal bones)

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Aristotle believed there are 3 kinds of events:

1. certain events
2. probable events
3. unpredictable events (i.e., those occurring in games of chance)

However, numerical patterns in games of chance were not analysed until 1000 AD.

The unfinished game

Pascal and Fermat, play a simple game: A fair coin is tossed up to five times. If the outcome is Head, Pascal wins 1 point. Otherwise, Fermat wins 1 point. The game stops when any of the players win 3 points. Whoever wins take the price of 100 coins. Assume that the current result is $2 : 1$ for Pascal and they need to stop playing the game. What is a fair way to distribute the 100 coins?

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Answer: 75 coins for Pascal and 25 for Fermat.

Without the concept of probability, this problem seemed to not be that elementary, as it is for us today.

In fact, this problem was known 150 years before Pascal and Fermat..

Prior to Pascal and Fermat

Some people we already discussed contemplated on the same question.

Luca Pacioli
(1447-1517)

Gerolamo Cardano
(1501-1576)

Niccolo Tartaglia
(1500-1557)

– Pacioli (a friend of Leonardo Da Vinci) first put the question in print, but with final goal - 6 points and score 5 : 2. He suggested (incorrectly) a division of 5 to 2 for the prize, according to the current score.

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- Cardano and Tartaglia corrected Pacioli, but did not get a solution.

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- Tartaglia believed that the unfinished game problem is impossible to solve.

Pascal

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- Pascal was born in central France. His father was a wealthy tax official and an amateur mathematicians.
- He never got university position, but did self-funded work in mathematics, natural science, religion on his own.
- In his twenties, he got ill (he was probably gluten intolerant) and did not recover until his dead.

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The Pascal's triangle and the binomial theorem

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\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 5 & & & & \\
 & & & 1 & & 10 & & 10 & & 5 & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 \\
 & 6 & & 1 & & 7 & & 21 & & 35 & \\
35 & & 21 & & 7 & & 1 & & & & \\
\end{array}
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The famous "Pascal's wager" (part of *Pensees*):

> *If God does not exist, one will lose nothing by believing in him, while if he does exist, one will lose everything by not believing.*

Fermat

Pierre de Fermat (1601-1665)

- Born in Bordeaux, south France.
- Studied law and then mathematics. He did not have any position as math professor, but spent immense amount of time on math and was one of the brightest!

Contributions of Fermat

Most famous with his work in number theory: the Fermat's last theorem, Little theorem of Fermat, Fermat's primes.

No one was able to prove Fermat's last theorem ($x^n + y^n \neq z^n$, when $n > 2$) for 400 years, until Andrew Wiles in 1994.

He also developed algebraic coordinate geometry independently of Descartes.

The correspondence

The first to bring the unfinished game problem to the attention of Pascal was a gambler (and an amateur mathematician) called Chevalier de Mere.

Pascal reached out to Fermat to ask whether his reasoning is correct.

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The first letters between Pascal and Fermat are lost. Only 7 of them are preserved. They never met in person, even though they were very close to each other.

Key takeaway

Determining the fair value of something is the central question in mathematics!

Fermat and Pascal confirmed this, as their problem turned out to be very improtant. Why?

First probability textbook

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He was the first to come up with the term *expected value* (expectation).

The 7 giants of statistics

1. **Jacob Bernoulli (1654 - 1705)**
   Famous Swiss mathematician. The law of large numbers.
2. **Abraham De Moivre (1667 - 1754)**
   French Hugenot, who escaped to England. Friend of Newton. Came up with generating functions and proved the CLT for binomial distribution.
3. **Pierre Simon Laplace (1749 - 1827)**
   From middle-class French to top scholar in Paris. First proof of a general CLT.
4. **Carl-Friedrich Gauss (1777 - 1855)**
   The prince of mathematics. The method of least squares (w/ Legendre).
5. **Karl Pearson (1857 - 1936)**
   Great organizer. Founded first statistics department.
6. **Ronald Fisher (1890 - 1962)**
   Multiple contributions transforming mathematical statistics.
7. **Jerzy Neyman (1894 - 1981)**
   Adopted Fisher's framework, brought it to continental Europe.

The law of large numbers

First formulation by Jacob Bernoulli (in “Ars Conjectandi”, 1713):

> *The probability of an event is morally certain to be approximated by the frequency with which it occurs!*

The name “Law of Large numbers” was given by Poisson much later.

The weak law of large numbers (WLLN)

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with a finite expected value $EX_i = \mu < \infty$. Then, for any $\epsilon > 0$,

$$\lim_{n \to \infty} P(|\overline{X} - \mu| \geq \epsilon) = 0.$$

The law of large numbers

Take 5000 chocolate bars. 2000 of them are dark, but you don't know this in advance.

How many of them should you sample (check out), to be sure that the observed frequency is within 2% of the true frequency (2=5), with very high probability (say 1=1000)?

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This was the question asked by Jacob Bernoulli. He showed that the observed frequency will get infinitely close to the true frequency, if we take big enough sample (i.e., the Law of Large Numbers).

Stated by Nicolaus Bernoulli (nephew of Jacob and Johann). Explanation was given by his cousin Daniel (the son of Johann).

The paradox

Toss a fair coin until you get an H.

If you toss H the first time, you get 2 coins. If, at the 2nd time - 4 coins, $\therefore$, if at the i-th time, you get 2^i coins.

How much coins you will pay to play the game?

The Central Limit Theorem

The same Nicolaus Bernoulli asked: If we know the true probability p and we sample 5000 chocolates, what is the chance that the observed number of dark chocolates lie in a given interval around its expectation $5000p$?

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Abraham de Moivre published a complete solution of the problem in his *The Doctrine of Chances* using calculus.

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The normal distribution is the only one with these properties!

Bibliography I

[1] http://www.cs.toronto.edu/~yuvalf/CLT.pdf