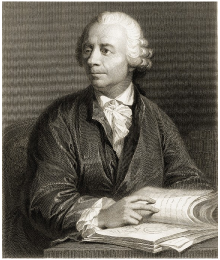
01:640:437 HISTORY OF MATHEMATICS

Stoyan Dimitrov

W8: Some results of Euler and Gauss

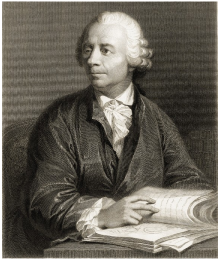
September 9, 2022

Euler



Leonard Euler (1707 - 1783)

– Born in Basel, Switzerland in the family of a Calvinist preacher.



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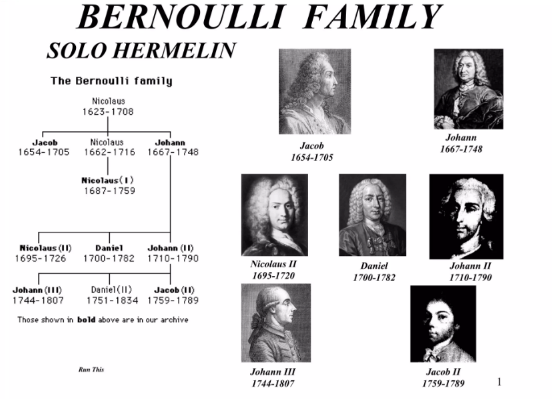
- His father arranged lessons of the little Euler with Johann Bernoulli.

Reminder: Who was Euler's teacher Johann Bernoulli?



Euler - early life

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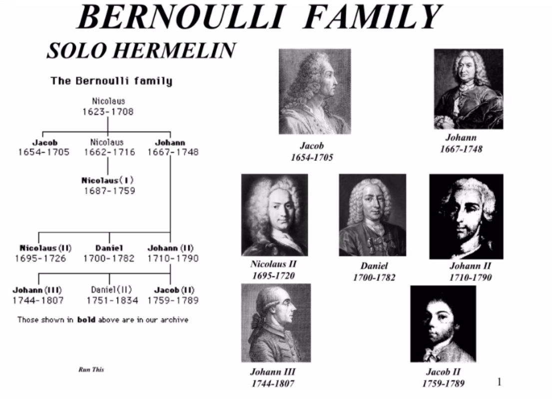

The grumpy J. Bernoulli soon recognized the talent of the young Leonard Euler.

Facts about Euler

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- Unlike many famous mathematicians, Euler enjoyed teaching.
- Euler lost the vision with one of his eyes quite early and his entire vision before he died.

Euler’s works

At some point Euler moved to Berlin, then went back to St.Peterburg, where he stayed until his dead.

Euler’s works:

- During his career, Euler published 886 books and articles in Latin, French and German!
- Roughly $1/3$ of all publications in math in the period $1725 - 1800$ were from Euler!

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- Roughly $1/3$ of all publications in math in the period $1725 - 1800$ were from Euler!
- $f(x)$, π, Σ and e are all notations introduced by Euler.
- Furthermore, we have *Euler constant*, *Euler triangle*, *Euler polynomial*, *Euler integral*, etc.

He found that

$$e^x = \sum_{x=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots . \tag{1}$$

The constant e was actually found by Jacob Bernoulli in 1683.
Why is e so important?

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e has some special properties:

- If c is a constant, for which $(c^x)' = c^x$, then $c = e$ (verify this using Equation (1)).
- If $\log_b x = \frac{1}{x}$, then $b = e$.
- J. Bernoilli found it as the value of $\lim_{n\to\infty}(1 + \frac{1}{n})^n$ (Why is this limit important?).

Famous results on series

Euler was the first to find the infinite sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Let us look how he did it..?

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First, what about the infinite sum (the so-called *harmonic series*):

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

There are many ways to prove that this sum is not finite (the harmonic series diverges).

What about the sum of the reciprocals of all primes

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Euler proved that this series is also divergent!:

$$\begin{aligned}
\log\left(\sum_{n=1}^{\infty} \frac{1}{n}\right) &= \log\left(\prod_p \frac{1}{1-p^{-1}}\right) = -\sum_p \log\left(1 - \frac{1}{p}\right) \\
&= \sum_p \left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \cdots\right) \\
&= \sum_p \frac{1}{p} + \frac{1}{2}\sum_p \frac{1}{p^2} + \frac{1}{3}\sum_p \frac{1}{p^3} + \frac{1}{4}\sum_p \frac{1}{p^4} + \cdots \\
&= A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \cdots \\
&= A + K
\end{aligned}$$

Let us look again at

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They just knew it is a number smaller than 2. Here is the brilliant solution of Euler ...

The $\pi^2/6$ series

1. First, take

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

   This "polynomial" must have roots $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, $\ldots$ (why?)

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2. Thus

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots =$$

$$\left[(1 - \tfrac{x}{\pi})(1 + \tfrac{x}{\pi})\right] \left[(1 - \tfrac{x}{2\pi})(1 + \tfrac{x}{2\pi})\right] \left[(1 - \tfrac{x}{3\pi})(1 + \tfrac{x}{3\pi})\right] \cdots =$$

$$\left[1 - \tfrac{x^2}{\pi^2}\right] \left[1 - \tfrac{x^2}{4\pi^2}\right] \left[1 - \tfrac{x^2}{9\pi^2}\right] \cdots$$

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3. If you foil the last parenthesis, you will see the coefficient in front of x^2 is $-(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots)$. This must be equal to $-\frac{1}{3!} = -\frac{1}{6}$.

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This is how we get $\frac{\pi^2}{6}$ for the sum!

Other series of Euler

Euler also found:

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24}.$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

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What about

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \quad = \quad ?$$

This is called the Apéry's constant $\zeta(3)$. By June 2020, more than 1.2 trillion of its digits after the decimal are known.

Euler's number theory

Proved or disproved many of Fermat's statements (Goldbach brought many of them to Euler's attention). Some examples:

– Every prime $p > 2$ of the kind $p = 4k + 1$ can be written as $a^2 + b^2$, for some unique a and b. If $p = 4k + 3$, no such representation exist!

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– Only 3 pairs of amicable numbers were known before Euler:
  $(220, 284)$ [The Ancient Greeks],
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- Fermat's little theorem: Every prime p divides $a^p - a$, where $gcd(a, p) = 1$.
- Euler refuted that $2^{2^n} + 1$ is always prime (by finding $2^{32} + 1 = 641 \times 6700417$).

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Carl Friedrich Gauss (1777-1855) on the Deutsche mark

– Born in Braunschweig, Central Germany in a poor working-class family.

– A child prodigy. Got sponsored by the Duke of Braunschweig to study in Braunschweig University of Technology and then in University of Göttingen.

Mathematical achievements

- The Normal distribution
- The method of least squares (w/ Legendre)
- The law of quadratic reciprocity (+ congruences)
- The EURIKA theorem (num = $\Delta + \Delta + \Delta$)
- The fundamental theorem of algebra

+ more ...

Mathematical achievements

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Until that moment people knew how to do it just for $n = 3, 5$ and 2^k sides. Gauss proved the following:

Theorem 1 (Gauss - Wantzel)

An n-gon is constructible if and only if $n = 2^k p_1 \cdots p_r$, where p_i are Fermat's primes (primes of the kind $2^{2^m} + 1$).

The Fundamental Theorem of Algebra

Theorem 2 (Gauss)

Every nonconstant polynomial with complex coefficients has a root in the complex numbers.

Example: A theorem unsuccesfully attempted by d'Alembert, Euler and others.

Gauss proved it in 1799. Later in his life, he gave 3 alternative proofs!

Later life of Gauss

When he was around 30, Gauss was appointed as a director of the Observatory at Göttingen. Then he worked on more applied problems.

Later, he encouraged Sophie Germain in her mathematical endeavours, even though this was not widely accepted at all!

The faith of Gauss?

When his son, Eugene, announced that he wanted to become a Christian missionary, Gauss approved of this, saying that regardless of the problems within religious organizations, missionary work was “a highly honorable task”.

Bibliography I