01:640:437 HISTORY OF MATHEMATICS

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W12: Selected events around 20th century

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Cantor

Georg Cantor (1845 - 1918)

- Cardinality of a set
- Diagonalization
- Continuum hypothesis
- Cantor’s set

Cardinality of a set

Cantor's Definition

Sets A and B have the same 'cardinality' (size), written |A| = |B|, if there exists a bijection between them.

E.g.: $|\mathbb{N}| = |Squares|$

because the function $f : \mathbb{N} \rightarrow Squares$ defined by $f(a)=a^2$ is a bijection.

Diagonalization

Cantor also showed that the real numbers $\mathbb{R}$ have bigger cardinality than the natural numbers $\mathbb{N}$:

Theorem: $\{0,1\}^\infty$ is NOT countable.

Consider the string formed by the 'diagonal':
the k-th bit in the (k-1)-st string

0: 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0...
1: 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1...
2: 1 0 1 1 0 1 1 1 0 1 1 1 1 0 1 1 1 1 1 1 0 1 1 1...
3: 0 0 1 1 0 1 0 1 0 0 0 1 0 1 0 0 0 1 0 1 0 0 0...
4: 0 1 0 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1...
5: 1 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0...

Continuum hypothesis (CH)

Is there a set with cardinality between those of $\mathbb{N}$ and $\mathbb{R}$?

CH cannot be proved or disproved from the standard axioms of set theory (Kurt Godel, 1940)

Cantor's set

Cantor Set is formed by repeatedly cutting out the open middle third of a line segment [0,1] (leaving end points) :

Cantor's set

How big is the Cantor's set? Let's see how much did we remove?

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$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots + \frac{2^{k-1}}{3^k} + \ldots = 1. \tag{1}$$

However, it turns out Cantor's set is uncountable!

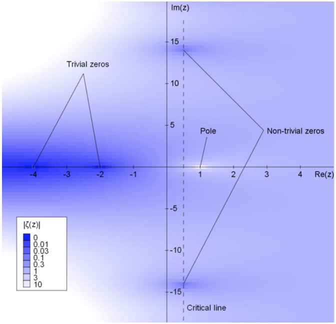
Hilbert's problems

In 1900, David Hilbert published a famous list of 23 math problems (kind of "the most wanted").

The most famous (and still unresolved is the Riemann Hypothesis (the 8th problem)).

Statement: $\zeta(s)$ has its zeros only at the negative even integers and complex numbers with real part $1/2$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$



Bibliography I

[1] https://www.andrew.cmu.edu/course/15-251/Lectures/lecture20.pdf