01:640:437 HISTORY OF MATHEMATICS

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W11: The development of Combinatorics and Algorithms.

April 12, 2023

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Combinatorics is the nanotechnology in mathematics.

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Branches of combinatorics:

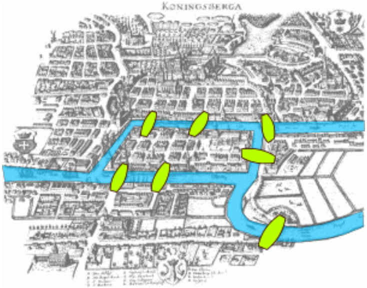
3.1 Enumerative combinatorics
3.2 Analytic combinatorics
3.3 Partition theory
3.4 Graph theory
3.5 Design theory
3.6 Finite geometry
3.7 Order theory
3.8 Matroid theory
3.9 Extremal combinatorics
3.10 Probabilistic combinatorics
3.11 Algebraic combinatorics
3.12 Combinatorics on words
3.13 Geometric combinatorics
3.14 Topological combinatorics
3.15 Arithmetic combinatorics
3.16 Infinitary combinatorics

Other Quotes about combinatorics: `https://www.math.ucla.edu/~pak/hidden/papers/Quotes/Combinatorics-quotes.htm`

and selection of my favorite of them: `https://stoyandimitrov.net/favoriteCombinatoricsQuotes.html`

Euler and the Königsberg Bridges

An important name in combinatorics: Leonard Euler



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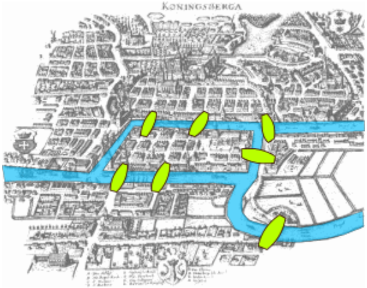

The problem of the Seven Bridges of Königsberg is a historically notable in combinatorics (and in mathematics).

Its negative resolution by Leonhard Euler in 1736 laid the foundations of graph theory and prefigured the idea of topology.

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Let’s see some examples from the first and most ancient branch of combinatorics (and math, in general):

Enumerative combinatorics! It is about counting things that are difficult to count!

The Twelvefold way

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This is known as the "Twelvefold way" and was introduced by Gian-Carlo Rota in mid 20th century (according to Richard Stanley).

The popular configurations

variations - Put n labeled balls into m labeled boxes (permutations, when $m = n$), with 1 balls in each box.

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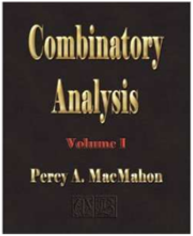
Some famous mathematicians working on partitions are Ramanujan and major Percy MacMahon.

Percy A. MacMahon, *Combinatory analysis*, Vol. 1, Cambridge University Press, 1915.



MacMahon is considered the most important person in enumerative combinatorics before 1960s.
In the book, MacMahon uses results of Cayley and Sylvester.

A lesser-known fact: MacMahon appears in the movie “The man who knew infinity”, where he outperforms Ramanujan on computational tasks.

MacMahon introduced the so-called *perfect* partitions. This is a partition, which contains exactly one partition of every smaller integer.

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Theorem 1 (MacMahon)

The number of perfect partitions of $n = p^k - 1$ *is* 2^{k-1}.

Ex. $7 = 2^3 - 1$ has $2^2 = 4$ perfect partitions, namely 7, $4 + 1 + 1 + 1$, $4 + 2 + 1$, $2 + 2 + 2 + 1$ and $1 + 1 + \cdots + 1$.

One result of Ramanujan on partitions

[Rogers-Ramanujan identity] The number of partitions of n into parts 1 mod 5 is equal to the number of partitions of n whose parts differ by at least 2.

Example:

$9 = 9 = 6 + 1 + 1 + 1 = 4 + 4 + 1 = 4 + 1 + 1 + 1 + 1 + 1 = 1 + \quad + 1$

$$9 = 9 = 8 + 1 = 7 + 2 = 6 + 3 = 5 + 3 + 1$$

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Important numbers in enumerative combinatorics, having many interpretations (R. Stanley has listed over 200 such in his book). Two of them are:

– Dyck paths - paths between $(0,0)$ and $(2n,0)$ with steps $(1,1)$ or $(1,-1)$, not going below the x-axis (the line $y=0$).

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- The number of ways to cut $(n+2)$-gon into triangles:

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Another important object in enumerative combinatorics.

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We can show that $d_n = (n-1)(d_{n-1} + d_{n-2})$ and that

$$n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}:$$

This is the integer closest to $\frac{n!}{e}$.

Tilings with dominos

Tiling problems are also an object of study in combinatorics.

Question 1

In how many ways can you tile $2 \times n$ rectangle with dominos (2×1 pieces)?

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$$f(n) = f(n-1) + f(n-2)$$

Given that $f(1) = 1$ and $f(2) = 2$,

we find $f(n) = \text{Fibonacci}(n+1)$

Tilings with dominos

What about tiling $3 \times n$ rectangles with dominos?

If the number of these tilings is $g(n)$, we can show that

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Finding a formula in the general case, when we $m \times n$ rectangle is more complicated.

Gian-Carlo Rota

Gian-Carlo Rota (1932 - 1999)

In 1960s, he wrote important papers establishing combinatorics, as a separate field. He also created the strongest combinatorics group at MIT.

Algorithms

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The modern notion of algorithms emerged in the UK in the mid 20th century with the development of computers!

Theoretical computer science - Branch of mathematics concerned with efficiency (say, in terms of time and memory), when performing a given computational tasks.

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Examples of algorithms we already studied:

- Euclid's algorithm for nding gcd.
- the Sieve of Eratosthenes.
- Archimedes algorithm for approximating

Does an algorithm always exist?

In 1900, David Hilbert proposed a list of 23 problems to be solved in the 20th century.

One of them was his tenth problem: Given a Diophantine equation (with any number of variables and integer coef cients), is there an algorithm telling you whether a solution exists? (people knew such for $x^n + y^n = z^n$)

De nition of Algorithm

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Roughly speaking, a primitive recursive function a function you can compute using a computer.

De nition of Algorithm

Then came the Turing machines (TMs) - abstraction of a computer.

They can compute the same set of primitive recursive functions.

The Halting problem

Alan Turing wanted to answer Hilbert's tenth problem.

Or in general: Is there a “mechanical process”, which can tell us whether any given mathematical statement is true or false?

The answer is NO (Turing proved it by showing that the HALTING problem is undecidable).

the P vs NP problem

Very important and still unresolved problem about algorithms.

The precise statement of the P versus NP problem was introduced in 1971 by Stephen Cook in his seminal paper "The complexity of theorem proving procedures" (and independently by Leonid Levin in 1973).

P and NP are 2 classes of problems. For the problems in P, we know fast algorithms and for those in NP we do not have such yet. However, we cannot show that $P \neq NP$, even though this is what most researchers believe.

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- Find whether two integers a and b are relatively prime?

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Two examples for problems in NP:

HAMPATH - Find whether a Hamilton path exists in a given graph?

CLIQUE - Find whether a given graph has a clique of size k?

Bibliography I

[1] https://www.math.ucla.edu/~pak/lectures/Slides/MMT-talk-UCLA.pdf