01:640:437 HISTORY OF MATHEMATICS

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W10: The development of Algebra.

April 10, 2022

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Answer: René Descartes (17th century)

Other notable names and findings:

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- Several other identities, i.e.,

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

and

$$1 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(The last one can be found in Chinese sources in early 14th century)

Algebra

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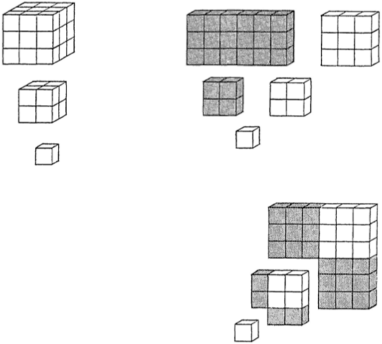
$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$$

Here is a proof without words for $n = 3$:



Other notable names and findings:

- the Persian mathematician Al-Khwarizmi: gave the first exhaustive explanation of the solution of a quadratic with positive roots.

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- Omar Khayyam (1050 – 1123 AD) looked at cubic equations!
- Tartaglia, Cardano, Ferrari: solution of the cubic and quartic.

More notable names and findings:

- Complex numbers appeared in the works of Bombelli (1572), Cardano and Descartes who called them “imaginary”, which meant to be derogatory.

  They were widely accepted after works of Euler and Gauss got published.

  Their geometric interpretation was given at the end of the 18th century.

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- Gauss proved the fundamental theorem of algebra (1799).
- Niels Abel proved that the general quintic is unsolvable in radicals.
- Then came the English mathematicians (Peacock, Boole, De Morgan, Hamilton, Cayley and Sylvester)

William Hamilton

William Rowan Hamilton (1805 - 1865)

William Hamilton knew 6 languages when he was 10. At age 22, he was appointed as a Royal Astronomer of Ireland.

William Hamilton

Prior to be 30 years old, Hamilton introduced a rule of multiplication of pairs of numbers (i.e., for the complex numbers):

$$(a, b)(\alpha, \beta) = (a\alpha - b\beta, a\beta + b\alpha)$$

He also interpreted that as a rotation in the plane.

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What if we use quadruples instead of triples of numbers and abandon the commutative law?!

Quaternions

A quaternion is an expression of the form

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

Where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the basic quaternions, satisfying

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

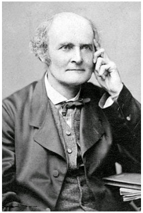
×	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

Hamilton stopped his walk, and, with a knife, he cut the fundamental formula $i^2 = j^2 = k^2 = ijk = -1$ on a stone of Brougham Bridge.

There is an organized walk, every year since 1990 on the same date, tracing the steps of Hamilton.

Hamilton spent the last 20 years working on the algebra of quaternions.

He wrote a book called *Lectures on Quaternions* (1853) including applications in geometry and physics.



Arthur Cayley (1821-1895)

A British mathematician who studied at Trinity college, Cambridge (later he taught there). He has many contributions to algebra and geometry.

One of the first to study matrices! He also introduced determinants.

Determinants

The determinant - the only function on square matrices, such that $|det(M)|$ has the properties:

1. Doing a row replacement on A does not change $|\det(A)|$.
2. Scaling a row of A by a scalar c multiplies $|\det(A)|$ by $|c|$.
3. Swapping two rows of a matrix does not change $|\det(A)|$.
4. The determinant of the identity matrix I_n is equal to 1.

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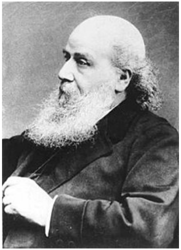
Theorem (Determinants and volumes). *Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbf{R}^n$, let P be the parallelepiped determined by these vectors, and let A be the matrix with rows $v_1, v_2, \ldots, v_n$. Then the absolute value of the determinant of A is the volume of P:*

$$|\det(A)| = \mathrm{vol}(P).$$



James Joseph Sylvester (1814 - 1897)

He coined the terms “discriminant” and “graph” in combinatorics.

Cayley and Sylvester

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Theorem 1 (Hamilton and Cayley)

Every square matrix is a root of its own characteristic polynomial.

Recall that the characteristic polynomial of a matrix A is $det(A - \lambda I)$. It helps to find the eigenvalues of the matrix.

Hamlton-Cayley theorem: Example

As a concrete example, let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Its characteristic polynomial is given by

$$p(\lambda) = \det(\lambda I_2 - A) = \det\begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix} = (\lambda - 1)(\lambda - 4) - (-2)(-3) = \lambda^2 - 5\lambda - 2.$$

The Cayley–Hamilton theorem claims that, if we *define*

$$p(X) = X^2 - 5X - 2I_2,$$

then

$$p(A) = A^2 - 5A - 2I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

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