# 01:640:437 HISTORY OF MATHEMATICS


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## W5: Cardano and the solution of the cubic

October 3, 2022

# The cubic equation



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Attention: The whole story of resolving this question is very peculiar!

You will hear it now!

# People involved in the solution of the cubic



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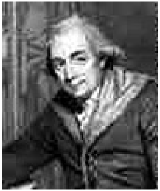
(1447-1517)



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(1500-1557)

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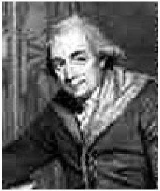

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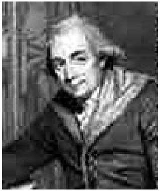

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Answer: Many Byzantine scholars sought refuge in the West (particularly Italy) after the invasion by the Ottoman empire.

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- we had, for example, people like Fibonacci (who popularized the Indo–Arabic numeral system by his “Liber Abaci”)
- Some Byzantine mathematicians designed Hagia Sophia (6th century):

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In 1494, in his *Summa de Arithmetica*, Luca Pacioli expressed doubts that the cubic can be solved at all (as with squaring of the circle problem).

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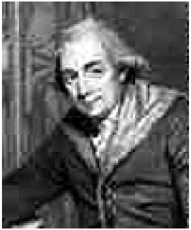

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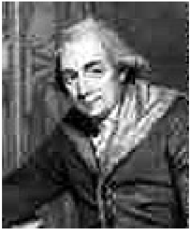

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Fun fact: Italians called this "cube and cosa equals number" ("cosa" = thing).

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At the last day of the challenge (actually the night of Febr 13, 1535), the exhausted Tartaglia found a solution!

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Cardano was the one to solve the cubic in its general form, using Tartaglia's depressed cubic solution!

# Cardano and his weird life

Cardano wrote his autobiography "The book of my life", full of peculiar stories. Here are selected facts from his life (based on that book):

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- Devoted Christian, but despite that, he was an adept astrologer, wore amulets, talked to spirits, etc..

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– He was jailed and died quietly in 1576.

# The final character in the story

So, what happened after Tartaglia revealed the secret of the depressed cubic to Cardano? Here is what Cardano said before that:

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Cardano shared Tartaglia's secret with his brilliant young servant Ludovico Ferrari who found a way to solve a general 4th degree polynomial equation!

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In 2 years, Cardano published Ars Magna (Great Art) containing solutions to equations of 3rd and 4th degree.

Tartaglia accused Cardano in theft, though Cardano acknowledged Tartaglia's work in the book.

This led to a public dispute between Tartaglia and Ferrari in Milan!

# The actual solution of the cubic



First, can we solve the following equations:

- $ax + b = 0$ (linear equation)
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Let $x = y - \frac{b}{2a}$. Then,

$$a(y-\frac{b}{2a})^2+b(y-\frac{b}{2a})+c = ay^2-by+by+\frac{b^2}{4a}-\frac{b^2}{2a}+c = ay^2+\frac{4ac-b^2}{4a} = 0.$$

What about the general cubic equation of Cardano:

$$ax^3 + bx^2 + cx + d = 0?$$

Well, similarly, plugging $x = y - \frac{b}{3a}$ reduces it to the depressed cubic

$$x^3 + mx + n = 0,$$

of Tartaglia! Why?

# Solving the depressed cubic



This is the more tricky part due to del Ferro and independently by Tartaglia:

Step 1: Think of $x$ as a difference of some variables $t$ and $u$. Look at the formula for $(t-u)^3$, that is,

$$(t-u)^3 = t^3 - 3t^2u + 3tu^2 - u^3.$$

Therefore, we know that for each $t$ and $u$:

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But this last system of 2 equations is equivalent to $t^3 - \frac{m^3}{27t^3} = n$ or to:

$$t^6 + nt^3 - \frac{m^3}{27} = 0.$$

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The expression in [] is a square if its discriminant is 0.
So, we shall solve a cubic equation, but we know how to do that!

# Things to remember from the story



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- What about polynomial equations of degree 5 and more?
  The Norwegian mathematician Niels Abel showed in the 19th century that there is no such formula in radicals for degree $\geq 5$!

# Bibliography I



[1] Dunham, W., 1991. Journey through Genius: Great Theorems of Mathematics. John Wiley  Sons.