Pointsets with bounded angles

Miroslav Marinov

CROPS, March 24th 2023

A natural start – the 90° and 60° regimes

Throughout we work in $\mathbb{R}^d$.

Problems

- What is the maximum number of points in $\mathbb{R}^d$ such that the angle formed by any three of them is at most 90°?

A natural start – the 90° and 60° regimes

Throughout we work in $\mathbb{R}^d$.

Problems

- What is the maximum number of points in $\mathbb{R}^d$ such that the angle formed by any three of them is at most 90°?
- What about at most 60°?

A natural start – the 90° and 60° regimes

Throughout we work in $\mathbb{R}^d$.

Problems

- What is the maximum number of points in $\mathbb{R}^d$ such that the angle formed by any three of them is at most 90°?
- What about at most 60°?
- What about *strictly less than* 90°?

A natural start – the 90° and 60° regimes

Throughout we work in $\mathbb{R}^d$.

Problems

- What is the maximum number of points in $\mathbb{R}^d$ such that the angle formed by any three of them is at most 90°?
- What about at most 60°?
- What about *strictly less than* 90°?

Corresponding answers:

- (Danzer and Grünbaum, 1962) 2^d, only hypercubes;

A natural start – the 90° and 60° regimes

Throughout we work in $\mathbb{R}^d$.

Problems

- What is the maximum number of points in $\mathbb{R}^d$ such that the angle formed by any three of them is at most 90°?
- What about at most 60°?
- What about *strictly less than* 90°?

Corresponding answers:

- (Danzer and Grünbaum, 1962) 2^d, only hypercubes;
- (Folklore) $d + 1$, regular d-simplex;

A natural start – the 90° and 60° regimes

Throughout we work in $\mathbb{R}^d$.

Problems

- What is the maximum number of points in $\mathbb{R}^d$ such that the angle formed by any three of them is at most 90°?
- What about at most 60°?
- What about *strictly less than* 90°?

Corresponding answers:

- (Danzer and Grünbaum, 1962) 2^d, only hypercubes;
- (Folklore) $d+1$, regular d-simplex;
- (Open, best bound by Gerenscér and Harangi, 2017) Between $2^{d-1}+1$ and 2^d-1. (An example of an unexpected *simple* breakthrough.)

Extreme acute regime – 61°

Denote by $f_\alpha(d)$ the maximum cardinality of a set of points in $\mathbb{R}^d$ such that all angles formed by three of them are *strictly less* than α.

Proposition (Erdős and Füredi, 1983)

For sufficiently small $c > 0$ we have

$$\left(1 + c^2\right)^d < f_{\frac{\pi}{3}+c}(d) < (1 + 4c)^d$$

Extreme acute regime – 61°

Denote by $f_\alpha(d)$ the maximum cardinality of a set of points in $\mathbb{R}^d$ such that all angles formed by three of them are *strictly less* than α.

Proposition (Erdős and Füredi, 1983)

For sufficiently small $c > 0$ we have

$$\left(1 + c^2\right)^d < f_{\frac{\pi}{3}+c}(d) < (1 + 4c)^d$$

(M., 2021) $(1 + c^2)^d$ can be improved up to $(1 + \Theta(c))^d$ with substantial technical difficulties.

Extreme acute regime – 61°

Idea: construct a large hypergraph and relate its edges to characteristic vectors of 0s and 1s – these can give points in $\mathbb{R}^d$.

Extreme acute regime – 61°

Idea: construct a large hypergraph and relate its edges to characteristic vectors of 0s and 1s – these can give points in $\mathbb{R}^d$.

Greedy large hypergraph

Fix $c \in (0,1)$ and positive integers k and d. There exists a k-uniform hypergraph G on $\{1, 2, \ldots, d\}$ with $|F_i \cap F_j| < ck$ for any distinct $F_i, F_j \in E(G)$ and

$$E(G) \geq \frac{\binom{d}{k}}{\sum_{j=\lceil ck \rceil}^{k} \binom{k}{j}\binom{d-k}{k-j}}.$$

Map F_i to v_i where m-th coordinate of v_i is 1 if $m \in F_i$ and 0 otherwise.

Extreme acute regime – 61°

Idea: construct a large hypergraph and relate its edges to characteristic vectors of 0s and 1s – these can give points in $\mathbb{R}^d$.

Greedy large hypergraph

Fix $c \in (0,1)$ and positive integers k and d. There exists a k-uniform hypergraph G on $\{1, 2, \ldots, d\}$ with $|F_i \cap F_j| < ck$ for any distinct $F_i, F_j \in E(G)$ and

$$E(G) \geq \frac{\binom{d}{k}}{\sum_{j=\lceil ck \rceil}^{k} \binom{k}{j}\binom{d-k}{k-j}}.$$

Map F_i to v_i where m-th coordinate of v_i is 1 if $m \in F_i$ and 0 otherwise.

We have $\sqrt{2k - 2ck} < |v_i - v_j| \leq \sqrt{2k}$.

Extreme acute regime – 61°

Idea: construct a large hypergraph and relate its edges to characteristic vectors of 0s and 1s – these can give points in $\mathbb{R}^d$.

Greedy large hypergraph

Fix $c \in (0,1)$ and positive integers k and d. There exists a k-uniform hypergraph G on $\{1, 2, \ldots, d\}$ with $|F_i \cap F_j| < ck$ for any distinct $F_i, F_j \in E(G)$ and

$$E(G) \geq \frac{\binom{d}{k}}{\sum_{j=\lceil ck \rceil}^{k} \binom{k}{j}\binom{d-k}{k-j}}.$$

Map F_i to v_i where m-th coordinate of v_i is 1 if $m \in F_i$ and 0 otherwise.

We have $\sqrt{2k - 2ck} < |v_i - v_j| \leq \sqrt{2k}$.

"The lower bound now follows by Stirling's formula".

A sphere packing relation with the upper bound

Rankin 1955

Fix $\alpha \in (0, \frac{\pi}{4})$ to be independent of d. Then the maximum possible number of pairwise disjoint hyperspherical caps in $\mathbb{S}^{d-1}$ does not exceed $f(\alpha)$, where

$$f(\alpha) \sim \frac{\sqrt{\frac{1}{2}\pi d^3 \cos 2\alpha}}{(\sqrt{2}\sin\alpha)^{d-1}}$$

Sketch: If largest distance is 1, then smallest is at least $\frac{\sin\left(\frac{\pi}{3}-2c\right)^2}{\sin\left(\frac{\pi}{3}+c\right)^2} > \frac{1}{\sqrt{2}}$ for small c by the Sine Law (applied to two triangles). By Jung's theorem there is a ball of radius $\frac{1}{\sqrt{2}}$ containing the points and a simple geometric argument shows that projecting them gives larger distances. Relate to angles and apply Rankin.

Thank you! Questions?