Lengths of Extremal, Irreducible, and Delicate Words

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Squares and overlaps

Definition

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"alfalfa" is an overlap, where $x =$ "a" and $Y =$ "lf".

Cubes

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A word is *squarefree* (respectively, *overlap-free*, *cubefree*) if it contains no squares (respectively, overlaps, cubes) as factors, i.e., contiguous subwords.

Avoiding squares and overlaps

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Theorem (Thue 1906)

There is an infinite squarefree ternary word. It begins 01202101...

Preview

	Ternary squarefree	Binary overlap-free	Binary cubefree
Extremal	✓ (Mol, Rampersad 2020)	✓ (Mol, Rampersad, Shallit 2020)	?
Irreducible	✓ (Harju 2020)	?	?
Delicate	?	?	?

Figure: Types of words and whether the lengths for which they exist are known

Preview (cont.)

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Irreducible	✓ (Harju 2020)	✓	✓
Delicate	✓	✓	✓

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A word is *extremal* with respect to a property if it satisfies the property, but inserting any letter from the alphabet into the word makes it no longer do so.

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Theorem (Grytczuk, Kordulewski, Niewiadomski 2020)

There are infinitely many extremal square-free ternary words.

Theorem (Mol, Rampersad 2020)

There is an extremal square-free ternary word of length n if and only if $n \in \{25, 41, 48, 50, 63, 71, 72, 77, 79, 81, 83, 84, 85\} \cup \{m \mid m \geq 87\}$.

Extremal words (cont.)

Theorem (Mol, Rampersad, Shallit 2020)

There is an extremal overlap-free binary word of length n if and only if $n \in \{10, 12\} \cup \{2k \mid k \geq 10\} \cup \{2^k + 1 \mid k \geq 5\} \cup \{3 \cdot 2^k + 1 \mid k \geq 3\}$.

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Conjecture (Grytczuk, Kordulewski, Niewiadomski 2020)

There are no extremal squarefree words over a four letter alphabet.

Extremal words (cont.)

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Question (implicit in Mol, Rampersad, Shallit 2020)

Are there any extremal cubefree binary words?

Irreducible words

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010212 is an irreducible squarefree ternary word.

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010212 is an irreducible squarefree ternary word.

Theorem (Harju 2020)

There is an irreducible squarefree ternary word of length n if and only if $n \in \{3, 6, 8, 9, 10, 11\} \cup \{m \mid m \geq 13\}$.

Irreducible words (cont.)

Theorem (P. 2021)

There is an irreducible overlap-free binary word of length n *if and only if* $n \in \{6, 8, 9, 10\} \cup \{m \mid m \geq 12\}$.

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There is an irreducible overlap-free binary word of length n if and only if $n \in \{6, 8, 9, 10\} \cup \{m \mid m \geq 12\}$.

Theorem (P. 2021)

There is an irreducible cubefree binary word of length n if and only if $n \in \{10, 14, 18, 19, 20\} \cup \{m \mid m \geq 22\}$.

Delicate words

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Theorem (P. 2021)

There is a delicate squarefree ternary word of length n *if and only if* $n \in \{5\} \cup \{m \mid m \geq 7\}$.

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Theorem (P. 2021)

There is a delicate overlap-free binary word of length n if and only if $n \in \{m \mid m \geq 7\}$.

Delicate words (cont.)

Theorem (P. 2021)

There is a delicate cubefree binary word of length n if and only if $n \in \{20, 21, 22, 29, 33, 34, 35\} \cup \{m \mid m \geq 38\}$.

Putting them all together

Theorem (P. 2021)

There are infinitely many extremal, irreducible, delicate overlap-free binary words.

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$$\begin{aligned} \mu(0) &= 01 \\ \mu(1) &= 10 \\ w_0 &= 011001101001100101100110100110 01 \\ w_{n+1} &= \mu(w_n). \end{aligned}$$

Future direction

Generalize delicate to k-delicate.

Definition

A word is *k-delicate* with respect to a property if it satisfies the property, but changing between 1 and k letters to other letters from the alphabet makes it no longer do so.

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Question

Are there finite k-delicate squarefree (respectively, overlap-free, cubefree) ternary (respectively, binary) words for all k?

Acknowledgments

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