Open Problems in Simple Graphs and Linear Hypergraphs

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Graph and a simple graph

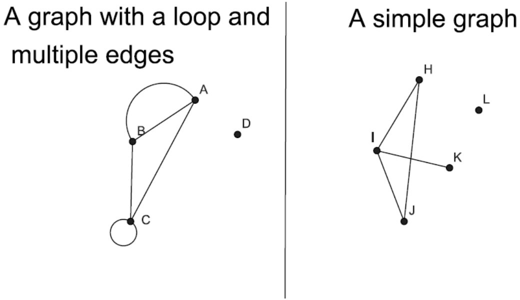
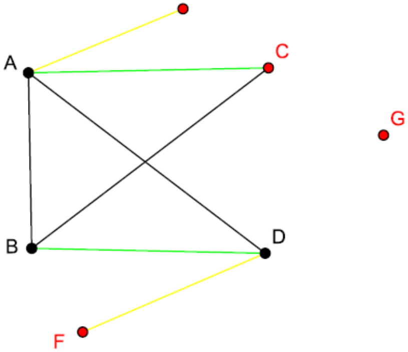


Figure: Is it simple ?

Independent set and Matching



Independence number and matching number

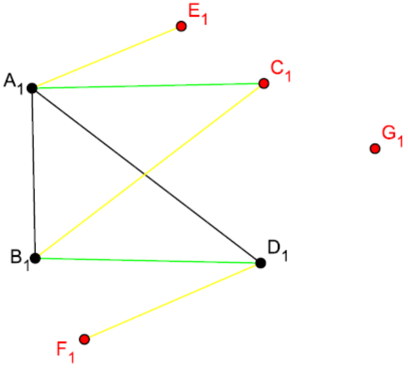


Figure: independence number 4 and matching number 3

Notation

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The maximum degree of G, denoted by $\Delta(G)$, is defined: $\Delta(G) := \max\{deg(x) \quad | \quad x \in V(G)\}$ where $deg(x)$ is the number of edges incident at x.

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The edge bound

Theorem (V. Chvátal and D. Hanson)

Let G be a simple graph such that $\Delta := \Delta(G)$ and $\nu := \nu(G)$. Then

$$|E(G)| \leq \Delta\nu + \left\lfloor \frac{\nu}{\lceil \frac{\Delta}{2} \rceil} \right\rfloor \left\lfloor \frac{\Delta}{2} \right\rfloor.$$

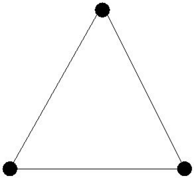

Figure: maximum degree 2, matching number 1

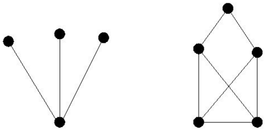

Figure: maximum degree 3, matching number 3

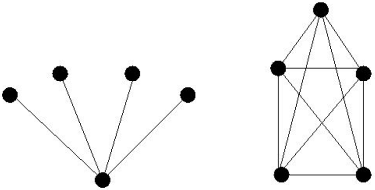

Figure: maximum degree 4, matching number 3

Unique extremal graphs

Theorem (N. Balachandran, K.)

Let G be a simple graph such that $\nu(G) \geq 2$, $\Delta(G) \geq 2$ and

$$|E(G)| = \Delta(G)\nu(G) + \left\lfloor \frac{\nu(G)}{\lceil \frac{\Delta(G)}{2} \rceil} \right\rfloor \lfloor \frac{\Delta(G)}{2} \rfloor.$$

G is a unique graph (up to isomorphism) if and only if $\lceil \frac{\Delta(G)}{2} \rceil$ divides $\nu(G)$.

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Erdős-Gallai edge bound I

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Erdős-Gallai edge bound II

Theorem (Erdős, Gallai)

Let G be a simple graph. If G has n vertices and matching size ν, then G has at most $\max\{\binom{2\nu+1}{2}, \binom{\nu}{2} + (n-\nu)\nu\}$ *edges.*

The original work of Erdős and Gallai can be found in [6]. A simpler proof of Erdős-Gallai bound was obtained by Akiyama and Frankl in [1].

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Definition

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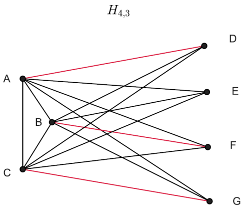
For $\alpha, \nu \in \mathbb{Z}^+$, we define a graph $H_{\alpha,\nu}$ as follows: $H_{\alpha,\nu}$: Start with the complete bipartite graph $K_{\alpha,\nu}$. Add $\binom{\nu}{2}$ edges to the part with ν vertices.
Note: $|E(H_{\alpha,\nu})| = \alpha\nu + \binom{\nu}{2}$, $\alpha(H_{\alpha,\nu}) = \alpha$, and, when $\alpha \geq \nu$, $\nu(H_{\alpha,\nu}) = \nu$.

Example $H_{4,3}$



Result I

[K., N. Mehta, N. Pulliyambalath]

Theorem

For all $\alpha, \nu \in \mathbb{Z}^+$, let G be a graph with $\alpha(G) \leq \alpha$ and $\nu(G) \leq \nu$.
(a) If $2\alpha < 3(\nu+1)$, then $|E(G)| \leq \binom{2\nu+1}{2}$ with equality holding iff G has $K_{2\nu+1}$ as one of its components.
(b) If $2\alpha > 3(\nu+1)$, then $|E(G)| \leq \alpha\nu + \binom{\nu}{2}$ with equality holding iff $G = H_{\alpha,\nu}$.
(c) If $2\alpha = 3(\nu+1)$, then $|E(G)| \leq \binom{2\nu+1}{2} = \alpha\nu + \binom{\nu}{2}$ with equality holding iff $G = H_{\alpha,\nu}$ or G has $K_{2\nu+1}$ as one of its components and other components are isolated vertices.

Result II (a)

[K., N. Mehta, N. Pulliyambalath]

Theorem

For all $n, \alpha, \nu \in \mathbb{Z}^+$, let G be a graph with $|V(G)| \leq n$, $\alpha(G) \leq \alpha$ and $\nu(G) \leq \nu$.
(a) If $n \leq 2\nu + 1$, then $|E(G)| \leq \binom{n}{2}$ with equality holding if and only if G is isomorphic to K_n.
(b) If $n > 2\nu + 1$ and $2\alpha < 3(\nu + 1)$, then $|E(G)| \leq \binom{2\nu+1}{2}$ with equality holding if and only if G has $K_{2\nu+1}$ as one of its components and other components are isolated vertices.
(c) If $n > 2\nu + 1$, $2\alpha > 3(\nu + 1)$ and $n \geq \alpha + \nu$, then $|E(G)| \leq \binom{\nu}{2} + \alpha\nu$ with equality if and only if G is isomorphic to $H_{\alpha,\nu}$.

Result II (b)

Theorem

(d) If $n > 2\nu + 1$, $2\alpha = 3(\nu + 1)$ *and* $n \geq \alpha + \nu$, *then* $|E(G)| \leq \binom{\nu}{2} + \alpha\nu = \binom{2\nu+1}{2}$ *with equality holding if and only if* G *is isomorphic to* $H_{\alpha,\nu}$ *or* G *has* $K_{2\nu+1}$ *as one of its components and other components are isolated vertices. .*

(e) If $n > 2\nu + 1$, $2\alpha \geq 3(\nu + 1)$ *and* $n < \alpha + \nu$, *then* $|E((G)| \leq \max\{\binom{2\nu+1}{2}, \binom{\nu}{2} + (n - \nu)\nu\}$ *with equality holding if and only if* G *has* $K_{2\nu+1}$ *as one of its components and other components are isolated vertices or* G *is isomorphic to* $H_{n-\nu,\nu}$*.*

Tools for graphs

- Augmenting path and maximum matching
- Removal of a vertex $\leftrightarrow |E(G \setminus x)| = |E(G)| - d_G(x)$ and creation of a new component $K_{1,\Delta}$.

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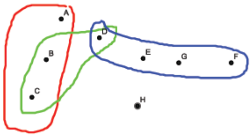
Application in Turán Type problems

- Dibek, Ekim, Heggernes, Maximum number of edges in claw-free graphs whose maximum degree and matching number are bounded, Discrete Mathematics. 2017 May 1;340(5):927-34.
- Wang, Hou , Liu , Ma, The Turán number for the edge blow-up of trees, Discrete Mathematics, 2021 Dec 1;344(12):112627.
- Yuan, Extremal graphs for edge blow-up of graphs. Journal of Combinatorial Theory, Series B, 2022 Jan 1;152:379-98.
- Belmonte , Heggernes , van't Hof, Rafiey, Saei, Graph classes and Ramsey numbers, Discrete applied mathematics, 2014 Aug 20;173:16-27.

Some open problems for simple graphs

- Finding maximum number of edges possible in a simple graph with restricted n, Δ, ν and α values and also determining which graphs achive the bound.
- Finding maximum number of edges possible in a simple graph with restricted n, Δ, ν values and also determining which graphs achive the bound.
- Finding maximum number of edges possible in a simple graph with restricted α, Δ, ν values and also determining which graphs achive the bound.
- Considering any of the previous Trań type problems from previous slides with three parameters for example n, ν and Δ or n, ν and α.

A hyper-graph



Sunflower

A *Sunflower* is a set system $\{A_1, \ldots, A_m\}$ along with a set X such that $A_i \cap A_j = X$ whenever $i \neq j$. The set X is called the core of the *Sunflower* and A_i's are called its petals.

An example of a Sunflower

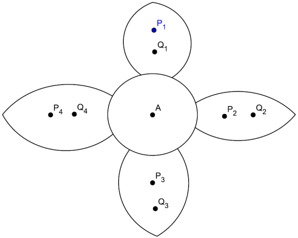


Figure: Sunflower with four petals

k-uniform

A non-empty family (or a set system) $\mathcal{F}$ is k-uniform if and only if $|A| = k$ for all $A \in \mathcal{F}$.

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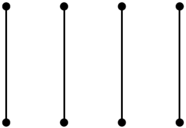
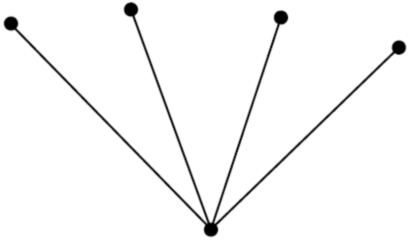
Sunflower in a graph I

A Sunflower in a simple graph with four petals (A)



Sunflower in a graph II

A Sunflower in a simple graph with four petals (B)



Existence of a Sunflower in a k-uniform family

Theorem (Erdős, Rado)

For any k-uniform set system, if $|\mathcal{F}| > k!(s-1)^k$ then $\mathcal{F}$ has a Sunflower with s petals.

The factor $k!$ is too large. The following is the best known improvement [2] is as follows:

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Restricted Erdős problem

Let $\mathcal{F}$ be a non-empty, *k*-uniform finite set system. Let $\Delta(\mathcal{F})$ be the maximum degree of any vertex in $X_{\mathcal{F}}$ and $\nu(\mathcal{F})$ be the size of a maximum matching. Note that a matching is a *Sunflower* with empty core. The problem is to find a bound on $|\mathcal{F}|$ in terms of $\Delta(\mathcal{F})$ and $\nu(\mathcal{F})$.

Some trivial bounds Let $\mathcal{F}$ be a *k*-uniform family. If $\Delta := \Delta(\mathcal{F})$, $\nu := \nu(\mathcal{F})$ then $|\mathcal{F}| \leq (\Delta - 1)(k\nu) + \nu = (k\Delta - (k-1))\nu$.

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Key Results

- 2-uniform hypergraphs, i.e., graphs
  - N. Balachandran, N. Khare, Graphs with restricted valency and matching number, **Discrete Mathematics**, 309(2009), 4176-4180,
- 3-uniform hypergraphs

Recalling trivial bound

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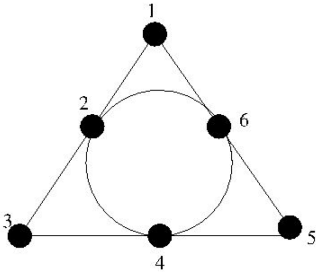

Figure: maximum degree 2, matching size 1, number of edges 3

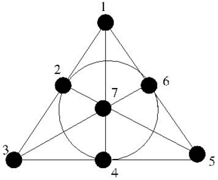

Figure: maximum degree 3, maximum matching 1

Hypergraph results I

A 3-uniform family $\mathcal{F}$ is called $\{0, 2\}$-intersecting family if and only if $|A \cap B| = 0$ or $|A \cap B| = 2$ for all $\{A, B\} \subseteq \mathcal{F}$.

We prove the following bound on the size of a $\{0, 2\}$-intersecting, 3-uniform family $\mathcal{F}$. If $\nu(\mathcal{F}) = \nu$ and $\Delta(\mathcal{F}) = \Delta$ then

$$|\mathcal{F}| \leq \begin{cases} 4\nu & \text{if } \Delta = 3; \\ \Delta\nu & \text{if } \Delta \neq 3. \end{cases}$$

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Hypergraph results II

A 3-homogeneous family $\mathcal{F}$ is called $\{1,2\}$-intersecting family if and only if $|A \cap B| = 1$ or $|A \cap B| = 2$ for all $\{A, B\} \subseteq \mathcal{F}$.

We show that the maximum size of a $\{1,2\}$-intersecting, 3-uniform family $\mathcal{F}$ has the following bound. Let $\Delta := \Delta(\mathcal{F})$. Then

$$|\mathcal{F}| \leq \begin{cases} \lfloor \frac{3\Delta}{2} \rfloor & \text{if } \Delta > 10; \\ 3\Delta - 2 & \text{if } \Delta \leq 10. \end{cases}$$

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Hypergraph results III

A k-homogeneous family $\mathcal{F}$ such that $|A \cap B| \leq 1$ for all $\{A, B\} \subseteq \mathcal{F}$ is called $\{0,1\}$-intersecting (also known as linear) family.

The following result appears in [9].

Theorem (K.)

Let $\mathcal{F}$ be a 3-uniform $\{0,1\}$-intersecting family with $\Delta = \Delta(\mathcal{F})$ and $\nu = \nu(\mathcal{F})$. If $\Delta \geq \frac{23}{6}\frac{\nu^2}{(\nu-1)} = \frac{67}{18}\nu(1+\frac{1}{\nu-1})$, then

Hypergraph results III

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$\mathcal{F}$ with $|\mathcal{F}| > \Delta\nu$: approach I example (a)

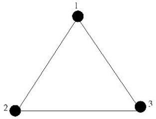
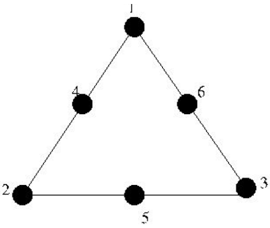


Figure: maximum degree 2, matching size 1, number of edges 3

$\mathcal{F}$ with $|\mathcal{F}| > \Delta\nu$: approach I example (b)



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Augmenting set : Let $\mathcal{F}$ be a hypergraph with a matching $\mathcal{M}$. We say $\mathcal{C} \subseteq \mathcal{F}$ is an $\mathcal{M}$-*augmenting set* if and only if $\mathcal{C}$ satisfies,

(1) $|\mathcal{M} \cap \mathcal{C}| < |\mathcal{C} \setminus \mathcal{M}|$
(i.e., there are more non-matching edges than matching edges in $\mathcal{C}$.)

(2) If $B \in \mathcal{M}$, $B \cap A \neq \emptyset$ for some $A \in \mathcal{C}$ then $B \in \mathcal{C}$.
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Some open problems for hypergraphs

- Improving bound on number of hyper edges in a 3-uniform linear hypergraph with restricted matching number and maximum degree
- Finding an asymptotic result on number of hyper edges in a k-uniform linear hypergraph as k goes to infinity with restricted matching number and maximum degree.
- Improving bound on existence of Sunflower in a k-uniform set system with bounds and tools developed in the step (ii) above (current best result [2])..

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